

# NUMERICAL APPROXIMATION OF TWO DIMENSIONAL SINGULARLY PERTURBED PROBLEMS

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To my parents and my family.

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## Abstract

We demonstrate how one can improve the numerical solution of singularly perturbed problems involving multiple boundary layers by using a combination of analytic and numerical tools. Incorporating the so-called boundary layer elements (BLE), which absorb the singularities due to the boundary layers, into finite element spaces can improve the accuracy of approximate solutions and result in significant simplifications. We discuss here convection-diffusion equations in the case where both ordinary and parabolic boundary layers are present.

We also revise the BLE so that it has a small compact support and hence the resulting linear system becomes sparse, more precisely, block tridiagonal. We prove the validity of the revised element for some singularly perturbed convection-diffusion equations via numerical simulations and via the  $H^1$ - approximation error analysis. Furthermore due to the compact structure of the BLE we are able to prove the  $L^2$ - stability analysis of the scheme and derive the  $L^2$ - error approximations.

## Contents

Chapter 1. Introduction	1
Chapter 2. Numerical Approximation of Multiple Boundary Layers	5
1. Introduction	5
2. Boundary Layer Analysis	10
3. Approximation via Finite Elements	30
4. A Mixed Boundary Value Problem	41
5. Occurrence of Boundary Layers	53
6. Numerical Simulations	56
7. Appendix	62
Chapter 3. $L^2$ -Stability Analysis	69
1. Introduction	69
2. Boundary Layer Elements (BLE)	72
3. $L^2$ - stability Analysis	76
4. $H^1$ - and $L^2$ - Approximation Errors	94
Bibliography	98

## CHAPTER 1

### Introduction

We consider singularly perturbed boundary value problems of the following type:

$$(0.1a) \quad -\epsilon \Delta u^\epsilon + bu_x^\epsilon + cu^\epsilon = f \quad \text{in } \Omega,$$

$$(0.1b) \quad u^\epsilon = 0 \quad \text{on } \partial\Omega,$$

where  $0 < \epsilon \ll 1$ ,  $b, c$  are constants with  $c \geq 0$ ,  $f = f(x, y)$  is smooth, and  $\Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$ . Other than the Dirichlet boundary conditions, various appropriate boundary conditions were imposed, e.g. Neumann, and channel flows (see [1] - [4], Chapters 2, 3).

As  $\epsilon$  becomes small, the solutions to such problems generally display near the boundaries thin transition layers called boundary layers, because the boundary conditions of the limit problem are different than those of the perturbed problem (0.1). As  $\epsilon \rightarrow 0$ , the functions  $u^\epsilon$  may display rapid variations near the boundary and their derivatives become very large. Resolving boundary layers is very costly in numerical computations and simulations. To motivate the study, we notice that the classical finite elements method is based on the following classical interpolation inequality: there exists an interpolant  $\Pi u^\epsilon \in V_N$ , where e.g.  $V_N$  is the span of the bilinear elements ( $Q_1$  elements), such that

$$(0.2) \quad \|u^\epsilon - \Pi u^\epsilon\|_{H^r(\Omega)} \leq \kappa \bar{h}^{2-r} |u^\epsilon|_{H^2(\Omega)}.$$

Here  $\kappa > 0$  is a constant independent of  $\bar{h}$ ,  $\epsilon$  and  $u^\epsilon$ ,  $r = 0, 1$ ,  $\bar{h} = \max\{h_1, h_2\}$ ,  $h_1, h_2$  are the mesh sizes respectively in the  $x$ -, and  $y$ - directions;  $|\cdot|_{H^s}$ ,  $\|\cdot\|_{H^s}$ ,  $s \geq 0$



integer, are the  $H^s$ - seminorm and the  $H^s$ - norm, respectively. By a priori estimates, it is not hard to find that

$$(0.3) \quad |u^\epsilon|_{H^2} \leq \kappa \epsilon^{-3/2} |f|_{L^2}.$$

Hence, for the interpolation inequality (0.2) to be effective, we have to require the mesh size  $\bar{h}$  to be of order e.g.  $o(\epsilon^{3/2})$  for  $r = 1$ , which is an *extremely fine mesh leading to very costly computations when  $\epsilon$  is small*. To remedy this difficulty we have used a variational approximation method combining analytic and numerical methods. Firstly we construct by asymptotic analysis the profile of the boundary layer. Based on this (or these) profile(s) we construct slightly simplified more numerically suitable versions of the profile(s), (an) adapted spline function(s) that we call the Boundary Layer Element(s) (BLEs). Our variational approach consists in adding these BLEs to the space  $V_N$  and thus obtaining a larger space  $\tilde{V}_N \supset V_N$ ; note that we do not use the expensive mesh refinement technique.

For problem (0.1), our goal was thus to construct a function  $g^\epsilon = g^\epsilon(x, y)$  which absorbs the  $H^2$ - singularity, namely,  $|u^\epsilon - g^\epsilon|_{H^2} \leq \kappa$ , so that for some  $\Pi(u^\epsilon - g^\epsilon) \in V_N$ ,

$$(0.4) \quad \|u^\epsilon - g^\epsilon - \Pi(u^\epsilon - g^\epsilon)\|_{H^r} \leq \kappa \bar{h}^{2-r} |u^\epsilon - g^\epsilon|_{H^2} \leq \kappa \bar{h}^{2-r};$$

after some modifications, the function  $g^\epsilon$  leading to the BLEs will be used in the numerical analysis and simulations. To derive the function  $g^\epsilon$  for the linear problems (0.1), we employ the technique of boundary layers and we analytically derive all the necessary boundary layers along the boundaries. We then slightly modify the boundary layers and obtain the BLEs which can be incorporated in the finite element spaces.

There are two boundary layers that affect the numerical computations which are the so-called ordinary and parabolic boundary layers. The ordinary boundary layers (OBL) are obtained by solving an ODE and they have the thickness of  $O(\epsilon)$  for (0.1).

If the discretization errors due to the OBLs are not properly handled, these errors propagate along the characteristics  $y = \text{constants}$ . For this reason, the classical numerical methods display wild oscillations. On the other hand, the parabolic boundary layers (PBL) are obtained by solving a parabolic equation (e.g. a heat equation) and they have the thickness of  $O(\sqrt{\epsilon})$  for (0.1). The discretization errors due to the PBLs are "localized" at the characteristics  $y = 0, 1$  because the errors arise parallel to the convective direction; they pollute the numerical solutions locally. In many cases, the PBLs can be handled independently of the discretization. The analysis of the PBLs is involved and developed in Chapter 2.

To handle the discretization errors due to the OBLs or the PBLs, let us firstly discuss the case where the OBLs are present in Eq. (0.1). Fortunately, the OBLs have a relatively simple form which can be computed easily in the discretizations. We have derived and revised the boundary layer elements (BLE) which are slightly modified from the OBLs so that they conform to the boundary conditions. More specifically, e.g. for the case where

$$(0.5) \quad b = 1, \quad c \geq 0,$$

we have a BLE at  $x = 0$ :

$$(0.6) \quad \phi_0(x) = -e^{-x/\epsilon} - (1 - e^{-1/\epsilon})x + 1,$$

or a more practical BLE which has a small compact support (see Figure 1):

$$(0.7a) \quad \phi_0^*(x) = \begin{cases} \tilde{\phi}_0(x) & \text{if } 0 \leq x \leq h_1, \\ 0 & \text{otherwise,} \end{cases}$$

with

$$(0.7b) \quad \tilde{\phi}_0(x) = -e^{-x/\epsilon} - (1 - e^{-h_1/\epsilon})(x/h_1) + 1.$$

Using the BLEs as in (0.6) and (0.7), we performed the numerical simulations for problem (0.1) with  $f$  smooth and a rectangle domain  $\Omega$  with the boundary conditions

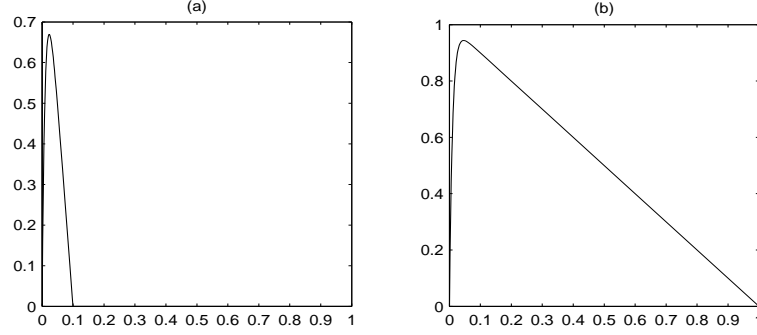


FIGURE 1. Boundary layer elements for  $\epsilon = 0.01$ ,  $h_1 = 0.1$  here; (a):  $\phi_0$ , (b):  $\phi_0^*$ .

of Dirichlet, Neumann (see [1] and Chapter 2 below), and Channel (see [2], [3]), and the results are extremely satisfactory. In particular we could avoid in such cases the use of the complicated and costly mesh refinements.

For the case where

$$(0.8) \quad b = 0, \quad c = 1,$$

we have only OBLs at the boundaries. Differently from the case (0.5), the OBLs have the thickness of  $O(\sqrt{\epsilon})$  for (0.1). Furthermore, we have the BLEs (0.6) and (2.3) with  $\epsilon$  being replaced by  $\sqrt{\epsilon}$ . The OBLs/BLEs appear at the four sides of  $\Omega$ . However, because of the absence of convective terms, the discretization errors due to these OBLs are localized. But these localized errors are overcome by these BLEs in the numerical computations and we actually obtained the numerical solutions with high accuracies (see [4]).

On the computational side, the BLE  $\phi_0^*(x)$  makes the discretized system in  $\tilde{V}_N$  for the linear problem (0.1) block tridiagonal; our problem can be solved very efficiently and it requires almost the same computing resources as the classical method which does not incorporate the BLE but with a much more accurate result for the same mesh. Furthermore, it turns out that the BLEs stabilize our discretized systems (see Chapter 3).

## CHAPTER 2

# Numerical Approximation of Multiple Boundary Layers

### 1. Introduction

In this chapter we consider linear singularly perturbed convection dominated boundary value problems of the following types:

$$(1.1a) \quad L_\epsilon u^\epsilon := -\epsilon \Delta u^\epsilon - u_x^\epsilon = f(x, y) \quad \text{for } (x, y) \in \Omega,$$

with boundary conditions

$$(1.1b) \quad u^\epsilon = 0 \quad \text{on } \partial\Omega,$$

or,

$$(1.1c) \quad \begin{aligned} u^\epsilon &= 0 \quad \text{at } x = 0, 1, \\ \frac{\partial u^\epsilon}{\partial y} &= 0 \quad \text{at } y = 0, 1. \end{aligned}$$

Here  $0 < \epsilon \ll 1$ , and  $\Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$ .

It can be shown (see below) that  $u^\epsilon \rightarrow u^0$  in  $L^2$  where  $u^0$  is the solution of the limit problem:

$$(1.2a) \quad -u_x^0 = f \quad \text{in } \Omega,$$

$$(1.2b) \quad u^0 = 0 \quad \text{at } x = 1,$$

so that we have

$$(1.2c) \quad u^0 = \int_x^1 f(s, y) ds.$$

Comparison between  $u^\epsilon$  and  $u^0$  is not easy because many discrepancies between  $u^\epsilon$  and  $u^0$  appear at the boundary. Just proving the  $L^2$ -convergence of  $u^\epsilon$  to  $u^0$  (which is

a byproduct of the analysis below) is not straightforward. For a comparison between  $u^\epsilon$  and  $u^0$  in smaller spaces (spaces of more regular functions), we need to introduce a number of boundary layers of different types to account for the discrepancies. The most common boundary layer appears at  $x = 0$  since  $u^0(0, y)$  does not vanish in general; this boundary layer is obtained using the technique of ordinary boundary layers (OBL). From (1.2c), we see also that some discrepancies appear in general at the boundaries  $y = 0, 1$ . These will be accounted for by a less common concept of boundary layer, namely the parabolic boundary layer (PBL).

In [1] we discussed the problem (1.1a), (1.1b) when  $f(x, y) = f_{yy}(x, y) = 0$  at  $y = 0, 1$ . In this case we only observe the discrepancy at  $x = 0$  (note that  $u^0(x, 0) = u^0(x, 1) = 0$ ), and the problem was thus handled by an OBL. In [3] we discussed equation (1.1a) in a channel with (1.1b) at  $y = 0, 1$  and periodicity in the  $x$ -direction; in this case we only observe parabolic boundary layers (PBL).

Here, by considering equation (1.1a) in a square, we theoretically and numerically investigate the case where both OBLs and PBLs are present. In fact some restriction (compatibility conditions) will be assumed on  $f$ ; indeed, as shown in [28], in the most general case (square with no restriction on  $f$ ), several other inconsistencies occur which have to be accounted for by still other boundary layers. In this chapter, as we said, we avoid these additional boundary layers, and consider cases where only OBLs and PBLs are present. In fact we will see that for the mixed boundary value problem (1.1a), (1.1c), the compatibility conditions on  $f$  and the effects of the PBLs are mild (see Section 4), whereas for (1.1a), (1.1b) we fully show how to overcome this compatibility condition issue.

Through the boundary layer analysis in Section 2, we will find rigorously that OBLs occur at the outflow  $x = 0$  and PBLs occur at the characteristic lines  $y = 0, 1$ . It turns out that OBLs and PBLs severely affect the numerical solutions because they

are, respectively, of order  $O(\epsilon^{-3/2})$  and  $O(\epsilon^{-3/4})$  in the  $H^2$ - norm. These  $H^2$ - singularities make our discretized (approximating) system highly unstable or ill-conditioned. Furthermore, if the boundary layers are not properly handled, the discretization errors due to the OBLs at  $x = 0$  pollute the whole domain  $\Omega$ , whereas the effect of the PBLs remain "localized" near the characteristic lines  $y = 0, 1$ . More precisely, the OBL errors propagate in the  $x$ - direction due to the convective term  $-u_x^\epsilon$  in (1.1) and hence if the discretization errors (the stiffness of the problem) are not properly accounted for, the approximate solutions display wild oscillations in the  $x$ - direction throughout the domain  $\Omega$  as in the classical approximation method, see Figure 2 (a) below and see also e.g. [9], [1], [2], [22], [25] and [27]. On the other hand, the discretization errors due to the PBLs at  $y = 0, 1$  are localized only at  $y = 0, 1$  because they are aligned parallel to the propagation direction  $x$ - axis, see Figure 3 (a) and 4 (a) below and see also e.g. [3], [25] and [27]. This phenomenon happens similarly in a reaction-diffusion problem in the absence of a convective term, see [4].

Our first aim in this chapter is thus to construct the ordinary and parabolic boundary layer elements (BLE) which, respectively, capture the singularities due to both OBLs and PBLs for the problems (1.1) under consideration. We numerically implement the BLEs in our approximating system, and thus we avoid the mesh refinements near the occurrences of each boundary layer and we are led to a significant simplification for the numerical implementations; we do not consider a (time-consuming) special mesh strategy and mesh refinement in the region of the boundary layers which are very costly in practice; we simply utilize a *uniform mesh*,  $Q_1$ - elements, that is the hat functions. See e.g. [12], [13], [18], [19], [20], [22], [23], [28] and [31] for many other developments on boundary layers and their asymptotic approximations, and see the book of [25] for the numerical aspects of singularly perturbed problems.

Because of the ordinary boundary layer (OBL) at  $x = 0$ , the approximate solutions are not stable in the  $H^1$  norm, but we do estimate the  $H^1$ - error for the approximate solution once the singular  $H^1$ - part has been captured using what we call below the boundary layer elements (BLE). However, it is noteworthy that our new discretized system (3.25) below is stable in the  $L^2$  space. More precisely, for any  $f \in L^2$

$$(1.3) \quad |u_N|_{L^2} \leq \kappa |f|_{L^2},$$

where  $u_N$  is an approximate solution obtained from (3.25), and a positive constant  $\kappa$  is independent of the mesh size  $\bar{h}$  and the small parameter  $\epsilon$ ; see the numerical results in the Tables 1 and 2 in [1] for a related situation. The  $L^2$ - stability analysis and  $L^2$ - error estimates for the current problem will appear in Chapter 3; in this case the analysis is technical due to the absence of a reaction term, e.g.  $u^\epsilon$ , in (1.1).

We would like to mention that during this research we improved the results of the chapter [1] in two respects. Firstly we could weaken the conditions needed to avoid the occurrence of PBLs for the Dirichlet boundary value problem. More precisely, if  $f = 0$  at  $y = 0, 1$ , PBLs do not exist in  $H^2(\Omega)$ , see Lemma 2.1 below. The second one is that we could weaken the compatibility conditions which appeared in [1] for the mixed boundary value problem, see (4.20) below.

We denote the mesh size by  $\bar{h} = \max\{h_1, h_2\}$  where  $h_1 = 1/M$ ,  $h_2 = 1/N$ ,  $M, N$  are the number of elements respectively in the  $x$ -, and  $y$ - directions. Hence, the number of rectangular elements is  $MN$ .

We shall consider the Sobolev spaces  $H^m(\Omega)$ ,  $m$  integer, equipped with the semi-norms,

$$(1.4) \quad |u|_{H^m} = \left( \sum_{|\alpha|=m} \int_{\Omega} |D^\alpha u|^2 dx dy \right)^{1/2},$$

the norms,

$$(1.5) \quad \|u\|_{H^m} = \left( \sum_{j=0}^m |u|_{H^j}^2 \right)^{1/2},$$

and the corresponding inner products,

$$(1.6a) \quad ((u, v))_{H^m(\Omega)} = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v),$$

where

$$(1.6b) \quad (u, v) = \int_{\Omega} uv dx dy.$$

We will also make use of a weighted energy norm which is useful when analyzing the convection-diffusion problem in the finite element context, namely:

$$(1.7) \quad \|u\|_{\epsilon} = \sqrt{\epsilon |\nabla u|_{L^2}^2 + |u|_{L^2}^2}.$$

As usual, when  $m = 0$ ,  $H^m$  is the space  $L^2$ . For the Dirichlet boundary value problem (1.1a), (1.1b), we use the Sobolev space  $H_0^1(\Omega)$ , which is the closure in the space  $H^1(\Omega)$  of  $C^\infty$  functions compactly supported in  $\Omega$ ; the appropriate space for (1.1a), (1.1c) will be introduced below. In the text  $\kappa$ ,  $c$  denote generic positive constants independent of  $\epsilon, h_1, h_2, \bar{h}$ , which may be different at different occurrences; the  $c$  are absolute constants, the  $\kappa$  are constants depending on the data.

We realize of course that the problem considered here is a model problem. A number of generalizations can be considered: more general elliptic operators, more general convection operators, nonlinear or time dependant problems; such generalizations will be considered elsewhere.

This chapter is organized as follows: we start in Section 2 by analyzing the boundary layers for the Dirichlet boundary value problem (1.1a), (1.1b) using asymptotic expansion techniques. We continue in Section 3 by constructing the boundary layer elements (BLE) via finite element methods which incorporate the BLEs and deriving



error estimates in  $H^1$ . In Section 4 we consider the mixed boundary value problem (1.1a), (1.1c) using a similar approach. It is important to identify the type of boundary layers that occur depending on the data  $f$  and the boundary conditions. In Section 5 we thus summarize two major boundary layers which are the OBLs and PBLs. Finally in Section 6 we show the numerical results that support our analysis.

## 2. Boundary Layer Analysis

Throughout this paper  $\Omega = (0, 1) \times (0, 1)$ , and  $f = f(x, y)$  is assumed to be smooth on  $\bar{\Omega}$ .

We first consider the Dirichlet boundary value problem (1.1a), (1.1b). Its weak formulation is as follows:

*To find  $u \in H_0^1(\Omega)$  such that*

$$(2.1a) \quad a_\epsilon(u, v) = F(v), \quad \forall v \in H_0^1(\Omega),$$

*where*

$$(2.1b) \quad a_\epsilon(u, v) = \epsilon \int_{\Omega} \nabla u \cdot \nabla v dx dy - \int_{\Omega} u_x v dx dy,$$

$$(2.1c) \quad F(v) = \int_{\Omega} f v dx dy.$$

It is easy to verify the coercivity of  $a_\epsilon$  on  $H_0^1(\Omega)$ , i.e.,

$$(2.2) \quad a_\epsilon(u, u) \geq \epsilon \|u\|^2, \quad \forall u \in H_0^1(\Omega),$$

the continuity of the bilinear form  $a_\epsilon$  on  $H_0^1 \times H_0^1$ , and the continuity of the linear form  $F$  on  $H_0^1$ . Hence, by the Lax-Milgram theorem, there exists a unique function  $u \in H_0^1(\Omega)$  satisfying equation (2.1).

Along the asymptotic analysis, we define the outer expansion  $u^\epsilon \sim \sum_{j=0}^{\infty} \epsilon^j u^j$ . By formal identification at each power of  $\epsilon$ , we find

$$(2.3a) \quad O(1) : -u_x^0 = f, \quad u^0 = 0 \text{ at } x = 1,$$

$$(2.3b) \quad O(\epsilon^j) : -\Delta u^{j-1} - u_x^j = 0, \quad u^j = 0 \text{ at } x = 1,$$

for  $j \geq 1$ . The boundary conditions in (2.3) are natural boundary conditions for the operator  $-d/dx$  on  $(0, 1)$ ; this choice of the boundary condition will be justified afterwards by the convergence theorem (see Theorem 2.2).

By explicit calculations, we find for  $j = 0, 1, 2$ :

$$(2.4a) \quad u^0(x, y) = \int_x^1 f(s, y) ds,$$

$$(2.4b) \quad u^1(x, y) = \int_x^1 \Delta u^0(s, y) ds = -f(1, y) + f(x, y) + \int_x^1 (s - x) f_{yy}(s, y) ds$$

$$(2.4c) \quad \begin{aligned} u^2(x, y) &= \int_x^1 \Delta u^1(s, y) ds = f_x(1, y) - f_x(x, y) - (1 - x) f_{yy}(1, y) \\ &\quad + 2 \int_x^1 f_{yy}(s, y) ds + \int_x^1 \frac{(s - x)^2}{2} \frac{\partial^4 f}{\partial y^4}(s, y) ds. \end{aligned}$$

**2.1. The Parabolic Boundary Layers.** It is clear that the functions  $u^j$  of the outer expansion do not generally satisfy the boundary conditions (1.1b) at  $x = 0$ , and  $y = 0, 1$ . To resolve these discrepancies, we will introduce the ordinary boundary layers (OBLs) (for  $x = 0$ ), and the so-called parabolic boundary layers (PBLs), for  $y = 0, 1$ . We start with the parabolic boundary layers which are defined by the inner expansion  $u^\epsilon \sim \sum_{j=0}^{\infty} \epsilon^j \varphi_l^j$  at  $y = 0$ , where  $\varphi_l^j = \varphi_l^j(x, \bar{y})$ ,  $\bar{y} = y/\sqrt{\epsilon}$ . Then we find:

$$-\sum_{j=0}^{\infty} \{ \epsilon^{j+1} \varphi_{lxx}^j + \epsilon^j \varphi_{l\bar{y}\bar{y}}^j \} - \sum_{j=0}^{\infty} \epsilon^j \varphi_{lx}^j = 0.$$

By formal identification at each power of  $\epsilon$ , we obtain the following heat equations in which  $-x$  is the timelike variable:

$$(2.5a) \quad \begin{aligned} O(1) : & \quad -\varphi_{l\bar{y}\bar{y}}^0 - \varphi_{lx}^0 = 0, \\ O(\epsilon^j) : & \quad -\varphi_{l\bar{y}\bar{y}}^j - \varphi_{lx}^j = \varphi_{lxx}^{j-1}, \text{ for } j \geq 1. \end{aligned}$$

The "initial" and boundary conditions that we choose (and that are justified below afterwards) are:

$$(2.5b) \quad \varphi_l^j(x, \bar{y}) = 0, \text{ at } x = 1,$$

$$(2.5c) \quad \varphi_l^j(x, 0) = r_j(x),$$

$$(2.5d) \quad \varphi_l^j(x, \bar{y}) \rightarrow 0 \text{ as } \bar{y} \rightarrow \infty,$$

where  $r_j(x) = -u^j(x, 0)$ ,  $j \geq 0$ .

**2.2. Construction and Properties of the  $\varphi_l^j$ .** Firstly, we consider the following heat equation in a semi-strip, see Theorem 20.3.1 in [8]. Let

$$(2.6) \quad D = \{(x, y) \in \mathbb{R}^2; 0 < x < 1, y > 0\}.$$

We are given  $f^*$  which is uniformly Hölder continuous in  $x$  and  $y$  for each compact subset of  $D$  and satisfies

$$(2.7) \quad |f^*(x, y)| \leq \kappa \exp(-\gamma y),$$

for some  $\gamma > 0$ , and all  $0 < x < 1$  and  $y > 0$ ; we are also given  $g^*$  which is continuous on  $[0, 1]$ . Then we look for  $u$  satisfying:

$$(2.8) \quad \begin{cases} -\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial y^2} = f^*, \text{ for } (x, y) \in D, \\ u(x, 0) = g^*(x), \text{ } 0 < x < 1, \\ u(x, y) \rightarrow 0 \text{ as } y \rightarrow \infty, \text{ } 0 < x < 1, \\ u(1, y) = 0. \end{cases}$$

**Compatibility Conditions.** We will assume the following smoothness and compatibility conditions on the data  $f^*$ ,  $g^*$  which guarantee that  $u \in C^l(\bar{D})$ ,  $l \geq 0$ , see e.g. [28], [29]:

$$(2.9a) \quad f^*(x, y) \text{ and } g^*(x) \text{ are sufficiently smooth}^1 \text{ on } \bar{D} \text{ and } [0, 1], \text{ respectively,}$$

and

$$(2.9b) \quad \frac{\partial^i}{\partial x^i} f^*(1, y) = \frac{\partial^i}{\partial x^i} g^*(1) = 0, \text{ for } 0 \leq i \leq l.$$

Let us recall the motivation for (2.9b): assume that  $u \in C^l(\bar{D})$  is a solution of (2.8), we then firstly notice that from the first equation (2.8), since  $\partial^2 u / \partial y^2 = -\partial u / \partial x - f^*$ , we recursively find

$$(2.10) \quad \begin{aligned} \frac{\partial^{2k}}{\partial y^{2k}} u &= \frac{\partial^{2(k-1)}}{\partial y^{2(k-1)}} \left( \frac{\partial^2}{\partial y^2} u \right) = -\frac{\partial^{2k-1}}{\partial x \partial y^{2(k-1)}} u - \frac{\partial^{2k-2}}{\partial y^{2(k-1)}} f^* \\ &= (-1)^k \frac{\partial^k}{\partial x^k} u + \sum_{s=0}^{k-1} (-1)^{s+1} \frac{\partial^{2k-s-2}}{\partial x^s \partial y^{2(k-s-1)}} f^*. \end{aligned}$$

Since  $u(1, y) = 0$ , setting  $x = 1$  in (2.10), we then easily find

$$(2.11) \quad (-1)^k \frac{\partial^k}{\partial x^k} g^*(1) + \sum_{s=0}^{k-1} (-1)^{s+1} \frac{\partial^{2k-s-2}}{\partial x^s \partial y^{2(k-s-1)}} f^*(1, y) = 0, \quad k = 0, 1, \dots, l,$$

which is necessary for  $u \in C^l(\bar{D})$ ; conditions (2.9) are much stronger than (2.11) and it is proven in e.g. [29] that the conditions (2.9) guarantee that  $u \in C^l(\bar{D})$ .

The case where these conditions (2.9) are not satisfied is more involved and will be considered elsewhere; we expect corner singularities at  $(1, 0)$ , see (7.6) below or see [28].

From now on we thus assume that the following compatibility conditions between the "initial" and boundary conditions for  $\varphi_l^j$  hold: for  $0 \leq i \leq 2n + d - 2j$ ,  $d = 0, 1$ ,

---

<sup>1</sup>The level of smoothness is unspecified for the sake of simplicity since smoothness is not the main issue in this chapter: we generally assume  $C^\infty(\bar{\Omega})$  regularity of the data, and from time to time we will mention weaker regularity assumptions which are sufficient; e.g. for (2.10), (2.15), we require  $f^* \in C^k(\bar{D})$  with  $k = \max\{2l - 2, l + 1\}$ .

and  $0 \leq j \leq n$ ,

$$(2.12) \quad r_j^{(i)}(1) = -\frac{\partial^i}{\partial x^i} u^j(1, 0) = 0.$$

To derive below the error estimate in the context of the standard finite elements method, we will need further estimates on the spatial derivatives of  $\varphi_l^j$ . To derive these estimates and for later purpose, it is useful to obtain the expression of the  $\varphi_l^j$  to be provided by the following lemma.

LEMMA 2.1. *Let  $u = u(x, y)$  be the solution of the heat equation (2.8) in  $D$ . Then the solution  $u$  is unique and it admits the integral representation:*

$$(2.13) \quad \begin{aligned} u(x, y) = & \sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) g^*\left(x + \frac{y^2}{2t^2}\right) dt \\ & + \frac{1}{2\sqrt{\pi}} \int_0^{1-x} \int_0^{\infty} \frac{1}{\sqrt{s}} \left\{ \exp\left[-\frac{(y-t)^2}{4s}\right] - \exp\left[-\frac{(y+t)^2}{4s}\right] \right\} f^*(x+s, t) dt ds, \end{aligned}$$

and

$$(2.14) \quad |u(x, y)| \leq \kappa \exp(-\gamma y), \text{ for the same } \gamma \text{ as in (2.7).}$$

If the conditions (2.9) hold, then  $u \in C^l(\bar{D})$ ,  $l \geq 0$ . Furthermore, if the following decay conditions hold:

$$(2.15) \quad \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} f^*(x, y) \right| \leq \kappa \exp(-\gamma y), \text{ for } 0 \leq i+m \leq l+1, \gamma > 0 \text{ as before,}$$

then the following pointwise estimates for  $u$  and its derivatives hold: for each  $i$  and  $m$ , there exists a constant  $\kappa_{im}$  which depends only on  $f^*$  and  $g^*$  such that

$$(2.16) \quad \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} u(x, y) \right| \leq \kappa_{im} \exp(-\gamma y), \forall (x, y) \in D,$$

for  $0 \leq i+m \leq l+1$ , the same  $\gamma$  as in (2.15).

For the proof, see the Appendix.

REMARK 2.1. From Lemma 2.1, if (2.9) and (2.15) hold, it is obvious that

$$(2.17a) \quad u \in C^l(\bar{D}) \cap H^{l+1}(D),$$

$$(2.17b) \quad u(x, \cdot) \in C^l([0, 1]) \cap H^{l+1}(0, 1), \quad \forall x \in [0, 1],$$

$$(2.17c) \quad u(\cdot, y) \in C^l([0, \infty)) \cap H^{l+1}(0, \infty), \quad \forall y \geq 0.$$

From Lemma 2.1 setting  $u = \varphi_l^k$  and  $y = \bar{y}$ , we find the solutions  $\varphi_l^j$  for equation (2.5) recursively:

$$(2.18a)$$

$$\varphi_l^0(x, \bar{y}) = \sqrt{\frac{2}{\pi}} \int_{\bar{y}/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) r_0\left(x + \frac{\bar{y}^2}{2t^2}\right) dt,$$

$$(2.18b)$$

$$\begin{aligned} \varphi_l^j(x, \bar{y}) &= \sqrt{\frac{2}{\pi}} \int_{\bar{y}/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) r_j\left(x + \frac{\bar{y}^2}{2t^2}\right) dt \\ &+ \frac{1}{2\sqrt{\pi}} \int_0^{1-x} \int_0^{\infty} \frac{1}{\sqrt{s}} \left\{ \exp\left[-\frac{(\bar{y}-t)^2}{4s}\right] - \exp\left[-\frac{(\bar{y}+t)^2}{4s}\right] \right\} \frac{\partial^2}{\partial x^2} \varphi_l^{j-1}(x+s, t) dt ds, \end{aligned}$$

for  $1 \leq j \leq n$ . Furthermore, thanks to the compatibility conditions (2.12), we find that for  $0 \leq j \leq n$ ,  $\varphi_l^j(x, \bar{y}) = \varphi_l^j(x, y/\sqrt{\epsilon})$  satisfies the regularities (2.17) with  $l = 2n + d - 2j$ , and  $y = \bar{y}$ .

The following lemmas easily follow from (2.16); the lemmas provide pointwise and norm estimates on the derivatives of  $\varphi_l^j$  which will be used below.

LEMMA 2.2. *Assume that the conditions (2.12) hold. Then there exist a positive constant  $\kappa_{ijm}$  independent of  $\epsilon$  such that the following inequalities hold*

$$(2.19) \quad \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \varphi_l^j \left( x, \frac{y}{\sqrt{\epsilon}} \right) \right| \leq \kappa_{ijm} \epsilon^{-m/2} \exp\left(-\frac{y}{\sqrt{\epsilon}}\right), \quad \forall (x, y) \in \bar{\Omega}^2,$$

for  $0 \leq i + m \leq 2n + d + 1 - 2j$ , and  $0 \leq j \leq n$ .

---

<sup>2</sup>  $\exp(-y/\sqrt{\epsilon})$  can be replaced by  $\exp(-cy/\sqrt{\epsilon})$  for any  $c > 0$  with then  $\kappa$  depending on  $c$ . Note that, by the estimate (2.16) applied to  $\varphi_l^0$  with  $f^* = 0$ , we find that  $\gamma > 0$  in (2.15) (and thus  $c > 0$ )

The following  $L^2$ - estimates are immediate consequences of Lemma 2.2.

LEMMA 2.3. *Assume that the conditions (2.12) hold. Let, for  $0 \leq \sigma < 1$ ,*

$$\Omega^\sigma = (0, 1) \times (\sigma, 1).$$

*Then there exists a positive constant  $\kappa_{ijm}$  independent of  $\epsilon$  such that the following inequalities hold: for  $0 \leq i + m \leq 2n + d + 1 - 2j$ ,*

$$(2.20a) \quad \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \varphi_l^j \right|_{L^2(\Omega^\sigma)} \leq \kappa_{ijm} \epsilon^{-m/2+1/4} \exp \left( -\frac{\sigma}{\sqrt{\epsilon}} \right);$$

*in particular,*

$$(2.20b) \quad \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \varphi_l^j \right|_{L^2(\Omega)} \leq \kappa_{ijm} \epsilon^{-m/2+1/4}.$$

REMARK 2.2. Similarly, at  $y = 1$ , we introduce another PBL,  $\varphi_u^j$ , having the same structure as  $\varphi_l^j$  with the role of  $\bar{y}$  and  $\tilde{y} = (1 - y)/\sqrt{\epsilon}$  being exchanged. We then need (assume) the following compatibility conditions, similar to (2.12):

$$(2.21) \quad -\frac{\partial^i}{\partial x^i} u^j(1, 1) = 0, \text{ for } 0 \leq i \leq 2n + d - 2j, d = 0, 1, \text{ and } 0 \leq j \leq n.$$

Under the hypothesis (2.21), the results of Lemma 2.2 and Lemma 2.3 are valid with  $\varphi_l^j$  replaced by  $\varphi_u^j$ ,  $\bar{y}$  by  $\tilde{y}$ , and  $(\sigma, 1)$  by  $(0, 1 - \sigma)$ . We also notice that  $\varphi_u^j(x, \tilde{y}) = \varphi_u^j(x, (1 - y)/\sqrt{\epsilon})$  satisfies the regularities (2.17) with  $l = 2n + d - 2j$ , and  $y = \tilde{y}$ . If  $\sigma = \kappa \epsilon^\alpha$  with  $\alpha < 1/2$ , the parabolic boundary layers,  $\varphi_l^j$  and  $\varphi_u^j$ , as indicated in the estimate (2.20), are exponentially small on  $(0, 1) \times (\sigma, 1 - \sigma)$ . This implies that we only need to take care of the parabolic boundary layers near the boundaries  $y = 0$  and  $y = 1$  in the finite element solutions.  $\square$

Before we go further, it is convenient here to recall the definition of exponentially small functions.

can be chosen arbitrarily. For  $j \geq 1$ , we apply (2.16) to  $\varphi_l^j$  with  $f^* = \partial^2 \varphi_l^{j-1} / \partial x^2$ , and then we find, recursively, that the same  $c$  is valid.

DEFINITION 2.1. A function  $\tilde{g}^\epsilon$  is called an exponentially small term, denoted *e.s.t.*, if there exists  $\alpha \in (0, 1)$  and  $\alpha' > 0$  such that for any  $k \geq 0$ , there exists a constant  $c_{\alpha, \alpha', k} > 0$  independent of  $\epsilon$  with

$$(2.22) \quad \|\tilde{g}^\epsilon\|_{H^k} \leq c_{\alpha, \alpha', k} e^{-\alpha/\epsilon^{\alpha'}}.$$

An *e.s.t.*( $n$ ) is a function  $\tilde{g}^\epsilon$  for which (2.22) holds for  $0 \leq k \leq n$ ;  $e^{-(1+x)/\epsilon}$  is an example of *e.s.t.*  $g(x)e^{-1/\epsilon}$  with  $g(x) = x \log x$  is an example of *e.s.t.*(1); note that  $g(x) \in H_0^1(0, 1)$  but  $g(x) \notin H^2(0, 1)$ .

**2.3. The Ordinary Boundary Layers.** At this stage the function  $u^\epsilon$  is tentatively approximated by  $\sum_{j=0}^\infty \epsilon^j (u^j + \varphi_l^j + \varphi_u^j)$ . However, with the definitions above of  $u^j$ ,  $\varphi_l^j$ , and  $\varphi_u^j$ , for each  $j$ ,  $0 \leq j \leq n$ , the function  $-u^j(x, y) - \varphi_l^j(x, \bar{y}) - \varphi_u^j(x, \tilde{y})$  is 0 at  $x = 1$  because of the boundary conditions (2.3), and (2.5b), and this function is exponentially small at  $y = 0$  and  $y = 1$  by the boundary conditions (2.5c) and Lemma 2.2. We now want to take care of the discrepancies at the boundary  $x = 0$  where

$$(2.23) \quad \begin{aligned} g^j(y) &= -u^j(0, y) - \varphi_l^j\left(0, \frac{y}{\sqrt{\epsilon}}\right) - \varphi_u^j\left(0, \frac{1-y}{\sqrt{\epsilon}}\right) \\ &\in C^{2n+d-2j}([0, 1]) \cap H^{2n+d+1-2j}(0, 1) \end{aligned}$$

does not vanish unlike  $u^\epsilon$ . We will handle these discrepancies with an ordinary boundary layer. Note that, in general, one cannot resolve these discrepancies with one single boundary layer, since while "repairing" the boundary condition at  $x = 1$ , we do not want to "damage" again the boundary conditions at  $y = 0, 1$ , which were "repaired" by the PBLs. In general, as we said above and in the Introduction, we cannot do this with one single boundary layer (see [28]); we can do this here because of the simplifying assumptions (2.12) and (2.21) which follow from (2.38) below.

For that purpose, we now introduce the so-called ordinary boundary layer functions  $\theta^j$  which are defined by the inner expansion  $u^\epsilon \sim \sum_{j=0}^\infty \epsilon^j \theta^j$  at  $x = 0$ , where



$\theta^j = \theta^j(\bar{x}, y)$ ,  $\bar{x} = x/\epsilon$ . By formal identification at each power of  $\epsilon$ , we find

$$(2.24a) \quad O(\epsilon^{-1}) : -\theta_{\bar{x}\bar{x}}^0 - \theta_{\bar{x}}^0 = 0,$$

$$(2.24b) \quad O(1) : -\theta_{\bar{x}\bar{x}}^1 - \theta_{\bar{x}}^1 = 0,$$

$$(2.24c) \quad O(\epsilon^{j-1}) : -\theta_{\bar{x}\bar{x}}^j - \theta_{\bar{x}}^j = \theta_{yy}^{j-2},$$

for  $2 \leq j \leq n$ . The boundary conditions are, for  $0 \leq j \leq n$ ,

$$(2.24d) \quad \theta^j = g^j(y), \text{ at } x = 0, \text{ and } \theta^j = 0, \text{ at } x = 1, g^j \text{ as in (2.23).}$$

By explicit calculations, we find: for  $j = 0, 1$ ,

$$(2.25a) \quad \begin{aligned} \theta^j \left( \frac{x}{\epsilon}, y \right) &= g^j(y) \left( \frac{e^{-x/\epsilon} - e^{-1/\epsilon}}{1 - e^{-1/\epsilon}} \right) \\ &= g^j(y) e^{-x/\epsilon} + e.s.t.(2n + d + 1 - 2j), \end{aligned}$$

and

$$(2.25b) \quad \begin{aligned} \theta^2 \left( \frac{x}{\epsilon}, y \right) &= g^2(y) \left( \frac{e^{-x/\epsilon} - e^{-1/\epsilon}}{1 - e^{-1/\epsilon}} \right) \\ &\quad + \epsilon^{-1} g_{yy}^0(y) \left( \frac{x(e^{-x/\epsilon} + e^{-1/\epsilon})}{1 - e^{-1/\epsilon}} - \frac{2e^{-1/\epsilon}}{1 - e^{-1/\epsilon}} \cdot \frac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}} \right) \\ &= g^2(y) e^{-x/\epsilon} + \epsilon^{-1} g_{yy}^0(y) x e^{-x/\epsilon} + e.s.t.(2n + d - 3). \end{aligned}$$

**2.4. Properties of the  $\theta^j$ .** We now derive the pointwise and norm estimates for the  $\theta^j$ . For that purpose, it is useful to obtain the expression of the  $\theta^j$ ,  $\forall j \geq 0$ , which is provided by the following lemma.

LEMMA 2.4. *We are given real numbers  $a, b, a > 0$ , and*

$$(2.26a) \quad f^* = f^{*,l}(x) = \sum_{n=0}^l f_n^* x^n \exp(-ax),$$

with  $f_n^* \in \mathbb{R}$ ,  $l \geq 0$  integer. Let  $u = u(x)$  be the solution of the ordinary differential equation in the region  $x > 0$ :

$$(2.26b) \quad -\frac{d^2u}{dx^2} - a\frac{du}{dx} = f^*, \quad x > 0,$$

$$(2.26c) \quad u(0) = b,$$

$$(2.26d) \quad u(x) \rightarrow 0 \text{ as } x \rightarrow \infty.$$

Then

$$(2.27) \quad u = u(x) = b \exp(-ax) + \sum_{n=0}^l u_n x^{n+1} \exp(-ax),$$

where the  $u_n \in \mathbb{R}$  are specified in the proof. Furthermore, we have the following pointwise estimates for  $u$  and its derivatives: for every  $i \geq 0$  and for any  $0 < c < a$ , there exist a positive constant  $\kappa_{il}$ , depending only on  $f^*$ ,  $c$  such that

$$(2.28) \quad \left| \frac{d^i}{dx^i} u(x) \right| \leq \kappa_{il} \exp(-cx).$$

PROOF. The solution to the homogeneous Eq (2.26b) (i.e., when  $f^* = 0$ ) is of the form:

$$(2.29) \quad u^h = u^h(x) = c_1 \exp(-ax) + c_2, \quad c_1, c_2 \in \mathbb{R}.$$

We then look for a particular solution of the nonhomogeneous Eq (2.26b):

$$(2.30) \quad u^p = u^p(x) = \sum_{n=0}^l u_n x^{n+1} \exp(-ax).$$

By substituting  $u^p$  for  $u$  in Eq (2.26b), we find the coefficients  $u_n$  recursively as follows:

$$(2.31) \quad \begin{cases} u_l = \frac{f_l^*}{a(l+1)} & \text{for } n = l, \\ u_n = a^{-1} \left\{ \frac{f_n^*}{n+1} + (n+2)u_{n+1} \right\} & \text{for } n = l-1, \dots, 0. \end{cases}$$

To comply with the boundary conditions, we set  $u = u^h + u^p$  and find  $c_1 = b$ ,  $c_2 = 0$  which leads to (2.27). The pointwise estimates (2.28) follow promptly from (2.27)

writing

$$(2.32) \quad \left| \frac{d^i}{dx^i} P(x) \right| \exp(-ax) \leq \kappa_{il} \exp(-cx),$$

where  $P(x)$  is a polynomial in  $x$  of degree  $\leq l$ , and  $c$  is any positive constant with  $c < a$ ,  $\kappa_{il} > 0$  is an appropriate constant depending on  $P(x)$  and  $c$ .  $\square$

REMARK 2.3. If instead of being constant  $f^* \in L^2(\mathbb{R}_+)$ , we set  $\tilde{u} = u - be^{-x}$ , where  $u$  is the solution of (2.26). Eq (2.26) is changed into:

$$(2.33) \quad \begin{cases} -\frac{d^2 \tilde{u}}{dx^2} - a \frac{d\tilde{u}}{dx} = f^* + (1-a)be^{-x} =: \tilde{f}^*, & x > 0, \\ \tilde{u}(0) = 0, \\ \tilde{u}(x) \rightarrow 0 \text{ as } x \rightarrow \infty. \end{cases}$$

Then  $\tilde{f}^* \in L^2(\mathbb{R}_+)$ , and from the Lax-Milgram theorem, there exists a unique solution  $\tilde{u} = u - be^{-x} \in H_0^1(\mathbb{R}_+)$  of Eq (2.33), and hence  $u \in H_0^1(\mathbb{R}_+)$ . In fact,  $\tilde{u}$  and thus  $u$  belong to  $H^2(\mathbb{R}_+)$ , and since  $H^1(\mathbb{R}_+) \subset C^{0,1/2}(\mathbb{R}_+)$ ,  $u$  is also in  $C^{1,1/2}(\mathbb{R}_+)$ .

Using Lemma 2.4, we now derive the pointwise and norm estimates for the OBLs in the following lemmas.

LEMMA 2.5. *Assume that the conditions (2.12) and (2.21) hold. For any  $0 < c < 1$ , there exist a positive constant  $\kappa_{ijm}$ , depending on  $c$  and on the data but independent of  $\epsilon$ , such that the following inequalities hold: for  $0 \leq i + m \leq 2n + d + 1 - 2j$ , and  $j = 2k$  or  $j = 2k + 1$  with  $k \geq 0$  integer, for all  $(x, y) \in \bar{\Omega}$ ,*

$$(2.34) \quad \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \theta^j \left( \frac{x}{\epsilon}, y \right) \right| \leq \kappa_{ijm} \epsilon^{-i} \exp \left( -c \frac{x}{\epsilon} \right) \left\{ 1 + \epsilon^{-k-m/2} \exp \left( -\frac{y}{\sqrt{\epsilon}} \right) + \epsilon^{-k-m/2} \exp \left( -\frac{1-y}{\sqrt{\epsilon}} \right) \right\} + e.s.t.(2n + d + 1 - 2j).$$

PROOF. We consider the following equation for  $\tilde{\theta}^j = \tilde{\theta}^j(\bar{x}, y)$ : for  $j = 2k$  with  $k \geq 0$  integer,

$$(2.35a) \quad -\tilde{\theta}_{\bar{x}\bar{x}}^j - \tilde{\theta}_{\bar{x}}^j = \tilde{\theta}_{yy}^{j-2}, (\tilde{\theta}^{-2} = 0 \text{ for convenience})$$

with the boundary conditions:

$$(2.35b) \quad \tilde{\theta}^j(\bar{x} = 0, y) = g^j(y), \quad \tilde{\theta}^j(\bar{x}, y) \rightarrow 0 \text{ as } \bar{x} \rightarrow \infty,$$

where  $g^j(y)$  is defined in (2.23). Then using Lemma 2.4 with  $x$  replaced by  $\bar{x}$  and  $u$  by  $\tilde{\theta}^j$ , we find the solutions  $\tilde{\theta}^j$  for  $j = 2k$  recursively, i.e.,  $\tilde{\theta}^j = P(\bar{x}, y) \exp(-\bar{x})$ , where  $P(\bar{x}, y)$  is a polynomial in  $\bar{x}$  of degree  $k$  whose coefficients are linear combinations of the  $\partial^{2s} g^{2k-2s}(y) / \partial y^{2s}$ ,  $s = 0, \dots, k$ . Hence, using the estimates in Lemma 2.2 and Remark 2.2, we find that for any  $0 < c < 1$ , there exist positive constants  $\kappa_{ijm}$  depending on  $c$  and the data but independent of  $\epsilon$  such that

$$\begin{aligned} \left| \frac{\partial^{i+m}}{\partial \bar{x}^i \partial y^m} \tilde{\theta}^j(\bar{x}, y) \right| &\leq \kappa_{ijm} \exp(-c\bar{x}) \max_{s=0, \dots, k} \left\{ \left| \frac{\partial^{2s+m}}{\partial y^{2s+m}} g^{2k-2s}(y) \right| \right\} \\ &\leq \kappa_{ijm} \exp(-c\bar{x}) \left\{ 1 + \epsilon^{-(2k+m)/2} \exp\left(-\frac{y}{\sqrt{\epsilon}}\right) + \epsilon^{-(2k+m)/2} \exp\left(-\frac{1-y}{\sqrt{\epsilon}}\right) \right\}. \end{aligned}$$

Comparing to Eq (2.24), we easily find that  $\theta^j(x, y) = \tilde{\theta}^j(\bar{x}, y) + e.s.t.(2n+d+1-2j)$ , hence the estimate (2.34) follows. For  $j = 2k + 1$ , the proof is similar.  $\square$

The following norm estimate is deduced immediately from Lemma 2.5.

LEMMA 2.6. For  $0 \leq \sigma_1, \sigma_2 < 1$ , let

$$\Omega^{\sigma_1, \sigma_2} = (\sigma_1, 1) \times (\sigma_2, 1 - \sigma_2).$$

Assume that the conditions (2.12) and (2.21) hold. Then, for any  $0 < c < 1$ , there exist a positive constant  $\kappa_{ijm}$ , depending on  $c$  and on the data but independent of  $\epsilon$ , such that the following inequalities hold: for  $0 \leq i + m \leq 2n + d + 1 - 2j$ , and  $j = 2k$

or  $j = 2k + 1$  with  $k \geq 0$  integer,

$$(2.36a) \quad \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \theta^j \right|_{L^2(\Omega^{\sigma_1, \sigma_2})} \leq \kappa_{ijm} \epsilon^{-i+1/2} \cdot \left( 1 + \epsilon^{-k-m/2+1/4} \exp \left( -\frac{\sigma_2}{\sqrt{\epsilon}} \right) \right) \exp \left( -c \frac{\sigma_1}{\epsilon} \right);$$

in particular,

$$(2.36b) \quad \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \theta^j \right|_{L^2(\Omega)} \leq \kappa_{ijm} \epsilon^{-i+1/2} (1 + \epsilon^{-k-m/2+1/4}).$$

REMARK 2.4. Let

$$(2.37a) \quad \sigma_1 = \kappa \epsilon^{\alpha_1} \text{ with } \alpha_1 < 1, \quad \sigma_2 = \kappa \epsilon^{\alpha_2} \text{ with } \alpha_2 < 1/2,$$

and

$$(2.37b) \quad \Omega_1 = (0, \sigma_1) \times \{(0, \sigma_2) \cup (1 - \sigma_2, 1)\}, \quad \Omega_2 = (0, \sigma_1) \times (\sigma_2, 1 - \sigma_2),$$

$$(2.37c) \quad \Omega_3 = (\sigma_1, 1) \times \{(0, \sigma_2) \cup (1 - \sigma_2, 1)\}, \quad \Omega_4 = (\sigma_1, 1) \times (\sigma_2, 1 - \sigma_2).$$

Then under the situation that OBLs and PBLs are present, as indicated in Lemma 2.6, we need to take care of the large variations of the derivatives due to the OBLs and PBLs in the subdomain  $\Omega_1$ , due to the OBLs in the subdomain  $\Omega_2$ , and, as indicated in Lemma 2.3 and Remark 2.2, due to the PBLs in the subdomain  $\Omega_3$ . Notice that the OBLs and PBLs are both exponentially small on  $\Omega_4$ .

**2.5. Asymptotic Error Analysis.** We conclude this study with the following theorems, which provide the asymptotic approximations and *which justify, on the theoretical side, the formal expansions we introduced before*. Below we focus on the  $H^2$ -asymptotic error which needs to be an  $O(1)$  quantity so as to absorb all  $H^2$ -singularities due to the boundary layers. This provides the justification for the construction of the boundary layer elements in the finite elements context, see Section 3. To avoid the singularities of the derivatives of PBLs at the vertices, we need the

following compatibility conditions on  $f$  (see (2.39) below):

$$(2.38a) \quad f(1, 0) = f_x(1, 0) = f_{xx}(1, 0) = 0,$$

$$(2.38b) \quad f(1, 1) = f_x(1, 1) = f_{xx}(1, 1) = 0.$$

From the explicit expressions of the  $u^j$  as in (2.4), we then have thanks to (2.38):

$$(2.39) \quad -\frac{\partial^i}{\partial x^i} u^j(1, 0) = -\frac{\partial^i}{\partial x^i} u^j(1, 1) = 0, \quad \text{for } 0 \leq i \leq 3 - 2j, j = 0, 1,$$

which are exactly the compatibility conditions (2.12) and (2.21); we note here that  $n = d = 1$ . Hence,  $\varphi_l^0, \varphi_u^0, \theta^0$  satisfy the regularity (2.17) with  $l = 3$  and  $\varphi_l^1, \varphi_u^1, \theta^1$  satisfy the regularity (2.17) with  $l = 1$ .

To obtain the asymptotic error estimate, we set

$$(2.40a) \quad w_{\epsilon n} = u^\epsilon - u_{\epsilon n} - \varphi_{l\epsilon n} - \varphi_{u\epsilon n} - \theta_{\epsilon n},$$

where

$$(2.40b) \quad u_{\epsilon n} = \sum_{j=0}^n \epsilon^j u^j, \quad \varphi_{l\epsilon n} = \sum_{j=0}^n \epsilon^j \varphi_l^j, \quad \varphi_{u\epsilon n} = \sum_{j=0}^n \epsilon^j \varphi_u^j, \quad \theta_{\epsilon n} = \sum_{j=0}^n \epsilon^j \theta^j.$$

We firstly notice that  $w_{\epsilon n}$  vanishes at  $x = 0, 1$  and hence, setting

$$(2.41) \quad \vartheta_{\epsilon n} = w_{\epsilon n}(x, 0)(1 - y) + w_{\epsilon n}(x, 1)y,$$

we find that  $W_{\epsilon n} := w_{\epsilon n} - \vartheta_{\epsilon n}$  satisfies the boundary condition (1.1b), namely  $W_{\epsilon n} = 0$  on  $\partial\Omega$ . We can here verify that  $\vartheta_{\epsilon n}$  is exponentially small. Indeed, from (2.41) and the explicit solution  $\theta^j$  in (2.25), and from Lemma 2.2, Remark 2.2 and Lemma 2.5, we find that for  $n = 0, 1$ ,

$$(2.42) \quad \begin{aligned} \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \vartheta_{\epsilon n} \right| &\leq \kappa \sum_{j=0}^n \epsilon^j \left\{ \left| \frac{\partial^i}{\partial x^i} \varphi_l^j(x, y=1) \right| + \left| \frac{\partial^i}{\partial x^i} \varphi_u^j(x, y=0) \right| \right. \\ &\quad \left. + (|\varphi_l^j(0, y=1)| + |\varphi_u^j(0, y=0)|) \left| \frac{\partial^i}{\partial x^i} e^{-x/\epsilon} \right| \right\} + \frac{\partial^{i+m}}{\partial x^i \partial y^m} (e.s.t.(2)) \\ &\leq \kappa(c) \exp\left(-c \frac{1}{\sqrt{\epsilon}}\right), \quad \text{for } 0 \leq i + m \leq 2, \text{ and for any } 0 < c < 1. \end{aligned}$$

From the outer expansion in (2.3), we have

$$(2.43) \quad -\epsilon \Delta u_{\epsilon n} - u_{\epsilon n x} = f - \epsilon^{n+1} \Delta u^n,$$

For the parabolic boundary layers defined in (2.5) and Remark 2.2, we have

$$(2.44a) \quad -\epsilon \Delta \varphi_{l\epsilon n} - \varphi_{l\epsilon n x} = -\epsilon^{n+1} \varphi_{lxx}^n,$$

$$(2.44b) \quad -\epsilon \Delta \varphi_{u\epsilon n} - \varphi_{u\epsilon n x} = -\epsilon^{n+1} \varphi_{u xx}^n,$$

and for the ordinary boundary layers defined in (2.24), we see that

$$(2.45) \quad -\epsilon \Delta \theta_{\epsilon n} - \theta_{\epsilon n x} = -\epsilon^n \theta_{yy}^{n-1} - \epsilon^{n+1} \theta_{yy}^n.$$

Subtracting (2.43), (2.44a), (2.44b), and (2.45) from (1.1a) and setting  $\theta^{-1} = 0$  for convenience, we write for  $W_{\epsilon n} = w_{\epsilon n} - \vartheta_{\epsilon n}$ ,

$$(2.46a) \quad L_\epsilon W_{\epsilon n} = -\epsilon \Delta W_{\epsilon n} - W_{\epsilon n x} = R^n + \tilde{R}^n \quad \text{in } \Omega,$$

with

$$(2.46b) \quad R^n = \epsilon^{n+1} \{ \Delta u^n + \varphi_{lxx}^n + \varphi_{u xx}^n + \theta_{yy}^n \} + \epsilon^n \theta_{yy}^{n-1}, \quad \tilde{R}^n = -L_\epsilon \vartheta_{\epsilon n}$$

and

$$(2.46c) \quad W_{\epsilon n} = 0 \quad \text{on } \partial\Omega;$$

note that thanks to (2.42), we easily see that the term  $\tilde{R}^n$  is exponentially small.

Then the asymptotic error estimates are provided in the next theorem and corollary.

Before we proceed, we mention the following simple regularity results, which will be used repeatedly later on, for Dirichlet or mixed boundary conditions:

LEMMA 2.7. *Let*

$$(2.47a) \quad V = H_0^1(\Omega) \quad \text{for (2.48b), or}$$

$$(2.47b) \quad V = \left\{ v \in H^1(\Omega); v = 0 \text{ at } x = 0, 1 \right\} \quad \text{for (2.48c).}$$

Let  $(V', \|\cdot\|_{V'})$  be the dual space of  $(V, \|\cdot\|_V)$ ,  $f^* = f^*(x, y) \in V'$ , and  $u$  the solution of equation:

$$(2.48a) \quad L_\epsilon u = -\epsilon \Delta u - u_x = f^* \text{ in } \Omega = (0, 1) \times (0, 1),$$

supplemented with either the boundary condition

$$(2.48b) \quad u = 0 \quad \text{on } \partial\Omega,$$

or with

$$(2.48c) \quad \begin{aligned} u &= 0 & \text{at } x = 0, 1, \\ \frac{\partial u}{\partial y} &= 0 & \text{at } y = 0, 1. \end{aligned}$$

Then the following regularity results hold.

If  $f^* = \epsilon f_1^* + f_2^*$  with  $f_1^* \in V'$ ,  $f_2^* \in L^2(\Omega)$ , then there exists a constant  $\kappa$  independent of  $\epsilon$  such that

$$(2.49) \quad \|u\|_\epsilon \leq \kappa \epsilon^{1/2} \|f_1^*\|_{V'} + \kappa |f_2^*|_{L^2(\Omega)},$$

and if  $f^* \in L^2(\Omega)$ ,

$$(2.50a) \quad \|u\|_\epsilon \leq \kappa |f^*|_{L^2(\Omega)},$$

$$(2.50b) \quad |u|_{H^2} \leq \kappa \epsilon^{-3/2} |f^*|_{L^2(\Omega)}.$$

PROOF. The weak formulation of the problem is as follows:  $u \in V$  and

$$(2.51a) \quad \tilde{a}_\epsilon(u, v) = \tilde{F}(v),$$

where

$$(2.51b) \quad \tilde{a}_\epsilon(u, v) = \epsilon \int_\Omega \nabla u \cdot \nabla v d\Omega - \int_\Omega u_x v d\Omega, \quad \tilde{F}(v) = \langle f^*, v \rangle;$$

if  $f^* \in L^2$ ,

$$(2.51c) \quad \langle f^*, v \rangle = (f^*, v) = \int_\Omega f^* v d\Omega.$$



Using elementary manipulations, and setting  $v = e^x u$  in (2.51a), we derive (2.49) from (2.51); (2.50a) is a particular case of (2.49). Finally, (2.50b) follows from (2.48a), observing that

$$(2.52) \quad |\Delta v|_{L^2}^2 = |v_{xx}|_{L^2}^2 + |v_{yy}|_{L^2}^2 + 2 \int_{\Omega} v_{xx} v_{yy} d\Omega,$$

and that, for both spaces  $V$ :

$$(2.53) \quad \int_{\Omega} v_{xx} v_{yy} d\Omega = \int_{\Omega} (v_{xy})^2 d\Omega, \quad \forall v \in V.$$

□

REMARK 2.5. We infer from (2.50b) that to make the solution  $u$  of equation (2.48) absorb the  $H^2$ - singularities, we will need an  $f^*$  of order  $\epsilon^{3/2}$  in  $L^2(\Omega)$ .

THEOREM 2.2. *Assume that the compatibility conditions (2.38) hold. Then for  $n = 0, 1$ ,*

$$(2.54a) \quad |w_{\epsilon n} - \delta_{\epsilon n}|_{L^2(\Omega)} \leq \kappa \epsilon^{n+3/4},$$

$$(2.54b) \quad \|w_{\epsilon n} - \delta_{\epsilon n}\|_{H^1(\Omega)} \leq \kappa \epsilon^{n+1/4},$$

$$(2.54c) \quad \|w_{\epsilon n} - \delta_{\epsilon n}\|_{H^2(\Omega)} \leq \kappa \epsilon^{n-3/4},$$

where the function  $\delta_{\epsilon n} \in H_0^1(\Omega)$  is specified in the proof and satisfies: for  $n = 0$ ,  $\delta_{\epsilon n} = 0$ , and for  $n = 1$ ,

$$(2.55) \quad |\delta_{\epsilon n}|_{L^2(\Omega)} \leq \kappa \epsilon^{3/4}, \quad \|\delta_{\epsilon n}\|_{H^1(\Omega)} \leq \kappa \epsilon^{1/4}.$$

PROOF. Firstly, we derive some estimates for  $R^n = R_1^n + R_2^n$  in the "error" equation (2.46). We write

$$(2.56a) \quad R_1^n = \epsilon^{n+1} \{ \Delta u^n + \varphi_{lxx}^n + \varphi_{uxx}^n + \theta_{yy}^n \},$$

$$(2.56b) \quad R_2^n = \epsilon^n \theta_{yy}^{n-1}.$$

Let  $\delta_{\epsilon n}$  be the solution of:

$$(2.57a) \quad L_\epsilon \delta_{\epsilon n} = R_2^n \quad \text{in } \Omega,$$

$$(2.57b) \quad \delta_{\epsilon n} = 0 \quad \text{on } \partial\Omega.$$

Then

$$(2.58a) \quad L_\epsilon(W_{\epsilon n} - \delta_{\epsilon n}) = R_1^n + \tilde{R}^n \quad \text{in } \Omega,$$

$$(2.58b) \quad W_{\epsilon n} - \delta_{\epsilon n} = 0 \quad \text{on } \partial\Omega.$$

From the norm estimates of Lemma 2.3 and Lemma 2.6, we find

$$(2.59) \quad \begin{aligned} |R_1^n|_{L^2} &\leq \epsilon^{n+1} \{ |\Delta u^n|_{L^2} + |\varphi_{lxx}^n|_{L^2} + |\varphi_{uxx}^n|_{L^2} + |\theta_{yy}^n|_{L^2} \} \\ &\leq \kappa \epsilon^{n+3/4}, \end{aligned}$$

$$|R_2^n|_{L^2} \leq \epsilon^n |\theta_{yy}^{n-1}|_{L^2} \leq \kappa \begin{cases} 0 & \text{for } n = 0, \\ \epsilon^{3/4} & \text{for } n = 1. \end{cases}$$

Furthermore, from (2.42), since  $\vartheta_{\epsilon n}$  and  $\tilde{R}^n$  are exponentially small, these terms can be absorbed in other norms and we may drop them. Then from Lemma 2.7 applied to equations (2.57) and (2.58) with  $f^* = R_2^n, R_1^n$ , respectively, the estimates (2.54) and (2.55) follow.  $\square$

If instead of (2.38) we impose the following stronger conditions (2.60) on  $f = f(x, y)$ :

$$(2.60) \quad f(x, 0) = f(x, 1) = 0,$$

we can remove the first parabolic boundary layers  $\varphi_l^0, \varphi_u^0$ , see Corollary 2.1 below. The parabolic boundary layers,  $\epsilon\varphi_l^1, \epsilon\varphi_u^1$ , are still present but they are very mild and their contributions are absorbed in the other  $H^2$ - terms. This is an improvement over the condition used in [1] to remove the parabolic boundary layers, that is

$$(2.61) \quad f(x, 0) = f_{yy}(x, 0) = f(x, 1) = f_{yy}(x, 1) = 0.$$

With the conditions (2.61), we proved in [1] that  $\|w_{\epsilon n}\|_{H^2(\Omega)} \leq \kappa$  for  $n = 1$ . We here prove the same result with the conditions (2.60), see Corollary 2.1 below.

**COROLLARY 2.1.** *Assume that the condition (2.60) holds. Then  $\varphi_l^0(x, y/\sqrt{\epsilon}) = \varphi_u^0(x, (1-y)/\sqrt{\epsilon}) = 0$  and for  $n = 0, 1$ ,*

$$(2.62a) \quad |w_{\epsilon n}|_{L^2(\Omega)} \leq \kappa \epsilon^{(3n+3)/4},$$

$$(2.62b) \quad \|w_{\epsilon n}\|_{H^1(\Omega)} \leq \kappa \epsilon^{(3n+1)/4},$$

$$(2.62c) \quad \|w_{\epsilon n}\|_{H^2(\Omega)} \leq \kappa \epsilon^{(3n-3)/4}.$$

**PROOF.** Because of (2.60), (2.4a) and (2.18a),  $\varphi_l^0(x, \bar{y}) = \varphi_u^0(x, \tilde{y}) = 0$ . The explicit expression of  $\theta^0$  in (2.25) then yields

$$(2.63) \quad \theta^0(x, y) = g^0(y)e^{-x/\epsilon} + e.s.t.(4) = -u^0(0, y)e^{-x/\epsilon} + e.s.t.(4).$$

Hence,

$$(2.64) \quad |R_2^n|_{L^2} \leq \epsilon^n |\theta_{\epsilon n}^{n-1}|_{L^2} \leq \kappa \begin{cases} 0 & \text{for } n = 0, \\ \epsilon^{3/2} & \text{for } n = 1, \end{cases}$$

and by (2.59),

$$(2.65) \quad |R^n| = |R_1^n + R_2^n|_{L^2} \leq |R_1^n|_{L^2} + |R_2^n|_{L^2} \leq \kappa \epsilon^{(3n+3)/4}.$$

Then from Lemma 2.7 applied to equation (2.46) with  $f^* = R^n = R_1^n + R_2^n$ , the lemma follows.  $\square$

**REMARK 2.6.** If we assume the conditions (2.38), from (2.54) with  $n = 0$ , using the norm estimates of Lemmas 2.3 and 2.6, we obtain

$$u^\epsilon = u^0 + O(\epsilon^{1/4}) \quad \text{in } L^2.$$

But if we assume the conditions (2.60), from Corollary 2.1 with  $n = 0$  and the estimate (2.63), we obtain

$$u^\epsilon = u^0 + O(\epsilon^{1/2}) \quad \text{in } L^2.$$

If we do not assume the compatibility conditions (2.38), we only obtain the following result at the order 0:

**THEOREM 2.3.** *For the Dirichlet boundary value problem (1.1a)-(1.1b), let the function  $f = f(x, y)$  be any smooth function on  $\bar{\Omega}$  (not necessarily satisfying (2.38)). Then*

$$(2.66a) \quad |u^\epsilon - u^0 - \varphi_l^0 - \varphi_u^0 - \theta^0|_{L^2(\Omega)} \leq \kappa \epsilon^{3/4},$$

$$(2.66b) \quad \|u^\epsilon - u^0 - \varphi_l^0 - \varphi_u^0 - \theta^0\|_{H^1(\Omega)} \leq \kappa \epsilon^{1/4}.$$

**PROOF.** We use equation (2.46) with  $n = 0$ , and since we do not require the compatibility conditions (2.38), we have

$$(2.67a) \quad L_\epsilon(W_{\epsilon 0}) = \epsilon R_1 + R_2 \quad \text{in } \Omega,$$

$$(2.67b) \quad W_{\epsilon 0} = 0 \quad \text{on } \partial\Omega,$$

where

$$(2.68a) \quad R_1 = \varphi_{lxx}^0 + \varphi_{uux}^0 + \theta_{yy}^0 - \epsilon^{-1} L_\epsilon \vartheta^0 \in H^{-1}(\Omega),$$

$$(2.68b) \quad R_2 = \epsilon \Delta u^0 \in C^\infty(\bar{\Omega}).$$

Note here that since  $-u^0(1, 0) = -u^0(1, 1) = 0$  from the explicit expression of  $u^0$  in (2.4), the compatibility conditions (2.12) and (2.21) hold with  $n = d = 0$ . We then find that  $\varphi_l^0, \varphi_u^0, \theta^0, \vartheta^0 \in H^1(\Omega)$ .

We are now able to find that

$$(2.69) \quad \|R_1\|_{H^{-1}} \leq \kappa \epsilon^{1/4}.$$

Indeed, for  $w \in H_0^1(\Omega)$ ,  $\|w\|_{H^1} = 1$ ,

$$\begin{aligned} \langle R_1, w \rangle &= \langle \varphi_{lxx}^0 + \varphi_{uux}^0 + \theta_{yy}^0 + \Delta \vartheta^0 + \epsilon^{-1} \vartheta_x^0, w \rangle \\ &= - \int_{\Omega} \left( \varphi_{lx}^0 w_x + \varphi_{ux}^0 w_x + \theta_y^0 w_y + \nabla \vartheta^0 \cdot \nabla w + \epsilon^{-1} \vartheta^0 w_x \right) d\Omega. \end{aligned}$$

Hence, from Lemmas 2.3 and 2.6,

$$(2.70) \quad | \langle R_1, w \rangle | \leq |\varphi_{lx}^0|_{L^2} + |\varphi_{ux}^0|_{L^2} + |\theta_y^0|_{L^2} \leq \kappa \epsilon^{1/4};$$

we dropped the exponentially small term  $\vartheta^0$  which is absorbed in other  $H^1$ - terms, and

$$(2.71) \quad \|R_1\|_{H^{-1}} = \sup_{\|w\|_{H^1}=1, w \in H_0^1} | \langle R_1, w \rangle | \leq \kappa \epsilon^{1/4}.$$

Hence, the theorem follows from Lemma 2.7 applied to equation (2.67) for  $W_{\epsilon_0}$  with  $f^* = \epsilon f_1^* + f_2^*$ ,  $f_1^* = R_1$ , and  $f_2^* = R_2$ , since  $\|f_1^*\|_{H^{-1}} \leq \|R_1\|_{H^{-1}} \leq \kappa \epsilon^{1/4}$  from (2.69) and  $|f_2^*|_{L^2} \leq |R_2|_{L^2} \leq \kappa \epsilon$ ; we here drop  $\vartheta^0$  too.  $\square$

REMARK 2.7. From Theorem 2.3 applied with  $n = 0$ , for any smooth function  $f$  which does not necessarily satisfy the compatibility conditions (2.38), using the norm estimates of Lemmas 2.3 and 2.6, we obtain that

$$u^\epsilon = u^0 + O(\epsilon^{1/4}) \quad \text{in } L^2.$$

### 3. Approximation via Finite Elements

In this section, we introduce the Boundary Layer Elements (BLEs) suitable for our problems, see  $\phi_0, \psi_l^n, \psi_u^n, \psi_0^n, \psi_N^n$  below; these elements are closely related to the correctors in the terminology of Lions [20]. We will show that these functions absorb the  $H^2$  singularity of  $u^\epsilon$ . We will incorporate  $\phi_0, \psi_l^n, \psi_u^n, \psi_0^n, \psi_N^n$  in the finite element spaces in which we will seek the approximate solutions; they will appear as special finite element functions - also called splines sometimes [26], [27].

**3.1. The Boundary Layer Elements : Constructions.** We now construct the Boundary Layer Elements (BLEs)  $\phi_0, \psi_l^n, \psi_u^n, \psi_0^n, \psi_N^n$  which will be shown by the two following lemmas to absorb the  $H^2$ - singularity of the solutions; we are interested

in two cases: the case where only *ordinary boundary layers* appear, and the case where *both ordinary and parabolic boundary layers* appear.

If the conditions (2.60) hold, then by Corollary 2.1 and Lemma 2.1 below,  $\varphi_l^0$  and  $\varphi_u^0$  do not appear, and, essentially, the singular terms only appear in the ordinary boundary layers  $\theta^j$ . Furthermore, we can extract the singular terms and slightly modify them so that they belong to the space  $V$ , that is, we derive the conforming ordinary boundary layer element:

$$(3.1) \quad \phi_0(x) = -e^{-x/\epsilon} - (1 - e^{-1/\epsilon})x + 1 \in H_0^1(0, 1);$$

we easily verify that

$$(3.2) \quad \|\phi_0\|_{H^m(0,1)} \leq \kappa(1 + \epsilon^{-m+1/2}), \quad \forall m \geq 0.$$

We will then approximate the exact solution  $u^\epsilon$  by the linear system (3.25) below.

If we do not impose the conditions (2.60), as in Theorem 2.3, the *parabolic boundary layers*  $\varphi_l^0$  and  $\varphi_u^0$  as well as the *ordinary boundary layer*  $\theta^0$  play an essential role in the singular behavior of the solutions. To capture the ordinary boundary layers, we will use the same  $\phi_0$  as in (3.1). We now construct the parabolic boundary layer elements which capture the functions  $\varphi_l^0$  and  $\varphi_u^0$ . Unlike for  $\phi_0$  we cannot extract simply the singular terms from  $\varphi_l^0$  and  $\varphi_u^0$ . Instead we start from the explicit expression of  $\varphi_l^0(x, \bar{y})$  and use it to construct the corresponding conforming parabolic boundary layer elements; the parabolic BLE at  $y = 1$  is then deduced by symmetry at the axis  $y = 1/2$ . In order to construct all the necessary BLEs, we thus decompose the data  $f$  as follows so that we can treat the OBLs and PBLs independently; we write:

$$(3.3a) \quad f(x, y) = f_1(x, y) + f_2(x, y) + f_3(x, y),$$

where

$$(3.3b) \quad f_1(x, y) = f(x, y) - f(x, 0)(1 - y) - f(x, 1)y,$$

$$(3.3c) \quad f_2(x, y) = \left( \sum_{j=0}^{n_0} \alpha_j x^j \right) (1 - y) + \left( \sum_{j=0}^{n_0} \beta_j x^j \right) y,$$

$$(3.3d) \quad f_3(x, y) = \left( f(x, 0) - \sum_{j=0}^{n_0} \alpha_j x^j \right) (1 - y) + \left( f(x, 1) - \sum_{j=0}^{n_0} \beta_j x^j \right) y.$$

The two polynomials in  $x$  appearing in (3.3c) are the Lagrange interpolating polynomials of degree  $n_0$  for  $f(x, 0)$  and  $f(x, 1)$  with nodes at the roots of the Chebyshev polynomial of degree  $n_0 + 1$ . Hence, we explicitly obtain the constants  $\alpha_j$  and  $\beta_j$  for  $j = 0, \dots, n_0$  from  $f(x, 0)$  and  $f(x, 1)$ , that is:

$$(3.4a) \quad \sum_{j=0}^{n_0} \alpha_j x^j = \sum_{j=0}^{n_0} f(x_j, 0) L_j(x), \quad \sum_{j=0}^{n_0} \beta_j x^j = \sum_{j=0}^{n_0} f(x_j, 1) L_j(x),$$

where

$$(3.4b) \quad L_j(x) = \prod_{i=0, i \neq j}^{n_0} \frac{(x - x_i)}{(x_j - x_i)},$$

with

$$(3.4c) \quad x_i = \frac{1}{2} \left( \cos \left( \frac{2i+1}{2(n_0+1)} \pi \right) + 1 \right).$$

For more details, see Lemma 3.3 and Corollary 3.1 below; see also (6.5d) - (6.5f).

For each  $j$ ,  $j = 1, 2, 3$ , we then consider the solution  $u_j = u_j^\epsilon$  of the following boundary value problem, particular case of (1.1):

$$(3.5a) \quad L_\epsilon u_j = f_j \quad \text{in } \Omega,$$

$$(3.5b) \quad u_j = 0 \quad \text{on } \partial\Omega;$$

as in (1.3) the weak formulation of this problem is:  $u_j \in H_0^1(\Omega)$  and

$$(3.6) \quad a_\epsilon(u_j, v) = (f_j, v), \quad \forall v \in H_0^1(\Omega).$$

For  $u_1$ , since  $f_1(x, 0) = f_1(x, 1) = 0$ , Corollary 2.1 and Lemma 2.1 below show that the parabolic boundary layers do not appear; only the OBLs are present. Hence, we will approximate  $u_1$  with the linear system (3.25) with  $f = f_1$  below.

For  $u_2$ , if  $f_2(x, 0) \not\equiv 0$  or  $f_2(x, 1) \not\equiv 0$ , some of the constants  $\alpha_j$  or  $\beta_j$  are nonzero, and hence due to the discrepancies between the outer solution, see (2.4a), and the boundary condition,  $u_2 = 0$  at  $y = 0, 1$ , we expect the presence of parabolic boundary layers as described before. To derive the functions  $\varphi_l^0(x, \bar{y})$  and  $\varphi_u^0(x, \tilde{y})$  corresponding to  $f_2$  and  $u_2$ , we first consider the case where  $f_2(x, 0)$  and  $f_2(x, 1)$  are the basic monomials

$$(3.7) \quad f_2(x, 0) = f_2(x, 1) = x^j.$$

We obtain  $\varphi_l^{0,j}(x, \bar{y})$  using the explicit expression in (2.18a), see (3.8e) below;  $\varphi_u^{0,j}(x, \tilde{y})$  is derived similarly. We then modify these functions to obtain the corresponding conforming BLEs,  $\psi_l^j, \psi_u^j$  belonging to  $H_0^1(\Omega)$ , and  $\psi_0^j, \psi_N^j$  belonging to  $H_0^1(0, 1)$ , that is

$$(3.8a) \quad \begin{aligned} \psi_l^j = \psi_l^j(x, y) &= \varphi_l^{0,j}(x, \bar{y}) + \varphi_l^{0,j}(0, \bar{y})(x - 1) \\ &\quad + (j + 1)^{-1}(x^{j+1} - x)(y - 1) + e.s.t.(1), \end{aligned}$$

$$(3.8b) \quad \psi_u^j = \psi_u^j(x, y) = \psi_l^j(x, 1 - y),$$

$$(3.8c) \quad \psi_0^j = \psi_0^j(y) = \varphi_l^{0,j}(0, \bar{y}) + (j + 1)^{-1}(1 - y) + e.s.t.(1),$$

$$(3.8d) \quad \psi_N^j = \psi_N^j(y) = \psi_0^j(1 - y),$$

where

$$(3.8e) \quad \varphi_l^{0,j}(x, \bar{y}) = \frac{\sqrt{2}}{\sqrt{\pi}(j + 1)} \int_{\bar{y}/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) \left[ \left(x + \frac{\bar{y}^2}{2t^2}\right)^{j+1} - 1 \right] dt.$$

See Lemma 3.2 below for the justification of these choices. Notice here that in (3.8) the  $e.s.t.(1)$  belong to  $H^1(\Omega)$  and to  $H^1(0, 1)$  respectively. They are introduced in the



analysis so that  $\psi_l^j, \psi_u^j$  belong to  $H_0^1(\Omega)$  and  $\psi_0^j, \psi_N^j$  belong to  $H_0^1(0, 1)$ , respectively, but they will be neglected in the numerical computations.

From Lemmas 2.2 and 2.3, we easily verify that, for  $m = 0, 1$ ,

$$(3.9a) \quad \|\psi_l^j(x, y)\|_{H^m(\Omega)}, \|\psi_u^j(x, y)\|_{H^m(\Omega)} \leq \kappa(1 + \epsilon^{-m/2+1/4}),$$

$$(3.9b) \quad \|\psi_0^j(y)\|_{H^m(0,1)}, \|\psi_N^j(y)\|_{H^m(0,1)} \leq \kappa(1 + \epsilon^{-m/2+1/4}).$$

We then obtain all the necessary parabolic boundary layer elements  $\varphi_l^{0,j}$  (and  $\varphi_u^{0,j}$ ),  $j = 0, \dots, n_0$  as in (3.8). We thus approximate  $u_2$  by  $u^*$  defined in (3.16b) below; the convergence errors are provided in Lemma 3.2 below.

Finally, for  $f_3$ , we truncate this function, that is the remainder of the Lagrange polynomial using all after  $n_0^{th}$  term; note that the error  $u_3$  due to  $f_3$  is independent of  $\epsilon$ . The precise error estimate in  $H^1$  will be given hereafter in Theorem 4.2.

The two following lemmas basically justify our constructions. They respectively prove that  $\phi_0$  absorbs the  $H^2$ - singularities, and they provide the asymptotic approximation error using the PBLs  $\varphi_l^{0,j}, \varphi_u^{0,j}$ , and the overlapping of the PBLs and OBLs. Lemma 2.1 will be used for  $f = f_1$  (and  $u^\epsilon = u_1$ ) and Lemma 3.2 will be used for  $f = f_2$  (and  $u^\epsilon = u_2$ ).

LEMMA 3.1. *Assume that*

$$(3.10) \quad f(x, 0) = f(x, 1) = 0, \quad \forall x \in [0, 1].$$

*Then there exist a positive constant  $\kappa$  independent of  $\epsilon$ , and a smooth function  $g = g^\epsilon(y) \in H_0^1(0, 1)$  with  $|g|_{H^2(0,1)} \leq \kappa$  such that*

$$(3.11) \quad \|u^\epsilon - g\phi_0\|_{H^2(\Omega)} \leq \kappa.$$

PROOF. We infer from Corollary 2.1, with  $n = 1$ , that

$$(3.12) \quad \|u^\epsilon - u^0 - \theta^0 - \epsilon \{u^1 + \varphi_l^1 + \varphi_u^1 + \theta^1\}\|_{H^2} \leq \kappa.$$

Since the  $u^j$  are independent of  $\epsilon$ , we find with Lemma 2.3,

$$(3.13) \quad \|u^\epsilon - \theta^0 - \epsilon\theta^1\|_{H^2} \leq \kappa,$$

and hence,

$$(3.14) \quad \|u^\epsilon + ge^{-x/\epsilon}\|_{H^2} \leq \kappa,$$

where

$$g = g^\epsilon(y) = u^0(0, y) + \epsilon(u^1(0, y) + \varphi_l^1(0, \bar{y}) + \varphi_u^1(0, \tilde{y})) + e.s.t.(2) \in H_0^1(0, 1).$$

Note that  $u^0(0, y) \in H_0^1(0, 1)$  because of (3.10). The role of the  $e.s.t.(2)$  is to make  $g$  belong to  $H_0^1(0, 1)$ . Then by Lemma 2.2, we easily find that  $|g|_{H^2(0,1)} \leq \kappa$ , and by the definition of  $\phi_0$  in (3.1), the lemma follows.  $\square$

LEMMA 3.2. *Assume that*

$$(3.15) \quad f(x, y) = \left( \sum_{j=0}^{n_0} \alpha_j x^j \right) (1 - y) + \left( \sum_{j=0}^{n_0} \beta_j x^j \right) y,$$

for some fixed constants  $\alpha_j, \beta_j \in \mathbb{R}$ , independent of  $\epsilon$ . Then there exist a positive constant  $\kappa(n_0)$  independent of  $\epsilon$  such that

$$(3.16a) \quad \|u^\epsilon - u^*\|_{H^m(\Omega)} \leq \kappa(n_0) \begin{cases} \epsilon^{3/4} & \text{for } m = 0, \\ \epsilon^{1/4} & \text{for } m = 1, \end{cases}$$

where

$$(3.16b) \quad u^* = \sum_{j=0}^{n_0} \alpha_j \psi_l^j + \sum_{j=0}^{n_0} \beta_j \psi_u^j + \sum_{j=0}^{n_0} \alpha_j \psi_0^j \phi_0 + \sum_{j=0}^{n_0} \beta_j \psi_N^j \phi_0.$$

PROOF. Let

$$(3.17) \quad f = f_l^j(x, y) = x^j(1 - y).$$

Then we easily find that  $\varphi_u^0 = 0$ , and from Theorem 2.3, we find

$$(3.18) \quad \|u_l^{\epsilon,j} - u^0 - \varphi_l^0 - \theta^0\|_{H^m} \leq \kappa(j)\epsilon^\gamma,$$

where  $\gamma = 3/4$ , or  $1/4$ , for  $m = 0$ , or  $1$ , respectively;  $u_l^{\epsilon,j}$  is the solution of Eq. (1.3) corresponding to  $f = f_l^j$  as in (3.17);  $u^0 = u_l^{0,j}$ ,  $\varphi_l^0 = \varphi_l^{0,j}$ ,  $\theta^0 = \theta_l^{0,j}$  are the corresponding outer solutions and boundary layer functions constructed as before. Notice that from the explicit expressions of  $u^0$ ,  $\theta^0$  in (2.4), (2.25), and from the expression of  $f$  in (3.17), we find

$$(3.19) \quad u^0(x, y) = \int_x^1 f(s, y) ds = (j+1)^{-1}(1-x^{j+1})(1-y);$$

in particular,

$$(3.20) \quad u^0(0, y) = (j+1)^{-1}(1-y).$$

Hence, we write

$$(3.21) \quad \begin{aligned} u^0 + \varphi_l^0 + \theta^0 &= u^0(x, y) + \varphi_l^0(x, \bar{y}) - (u^0(0, y) + \varphi_l^0(0, \bar{y}))e^{-x/\epsilon} + e.s.t.(1) \\ &= \psi_l^j(x, y) + \psi_0^j(y)\phi_0(x) + e.s.t.(1) \in H_0^1(\Omega), \end{aligned}$$

where  $\phi_0$ ,  $\psi_l^j$ ,  $\psi_0^j$  are defined in (3.1), (3.8a) - (3.8d). We then infer from (3.18), for  $m = 0, 1$ , that

$$(3.22) \quad \|u_l^{\epsilon,j} - \psi_l^j(x, y) - \psi_0^j(y)\phi_0(x)\|_{H^m} \leq \kappa(j)\epsilon^\gamma.$$

By considering the symmetry at the axis  $y = 1/2$ , we can deduce similar estimates for the solution  $u_u^{\epsilon,j}$  corresponding to the data  $f = f_u^j(x, y) = x^j y$ . Finally, by linearity and superposition of the solutions, we see that, for  $f$  as in (3.15),

$$(3.23) \quad u^\epsilon = u^{\epsilon, n_0} = \sum_{j=0}^{n_0} \alpha_j u_l^{\epsilon,j} + \sum_{j=0}^{n_0} \beta_j u_u^{\epsilon,j},$$

and

$$\begin{aligned} \|u^\epsilon - u^*\|_{H^m(\Omega)} &\leq \sum_{j=0}^{n_0} \alpha_j \|u_l^{\epsilon,j} - \psi_l^j(x, y) - \psi_0^j(y)\phi_0(x)\|_{H^m(\Omega)} \\ &+ \sum_{j=0}^{n_0} \beta_j \|u_u^{\epsilon,j} - \psi_u^j(x, y) - \psi_N^j(y)\phi_0(x)\|_{H^m(\Omega)} \leq \kappa(n_0)\epsilon^\gamma, \end{aligned}$$

where  $\gamma$  is defined in (3.18); the estimate (3.16) follows from (3.22) and the similar bound at  $y = 1$ .  $\square$

REMARK 3.1. In Lemmas 2.1 and 3.2, the term  $g\phi_0$  is due to the OBLs, whereas  $\sum_{j=0}^{n_0} \alpha_j \psi_l^j + \sum_{j=0}^{n_0} \beta_j \psi_u^j$  is due to the PBLs, and  $\sum_{j=0}^{n_0} \alpha_j \psi_0^j \phi_0 + \sum_{j=0}^{n_0} \beta_j \psi_N^j \phi_0$  is due to the PBLs and OBLs. Figure 1 gives the graphs of the boundary layer elements (3.1) and (3.8a) - (3.8d).

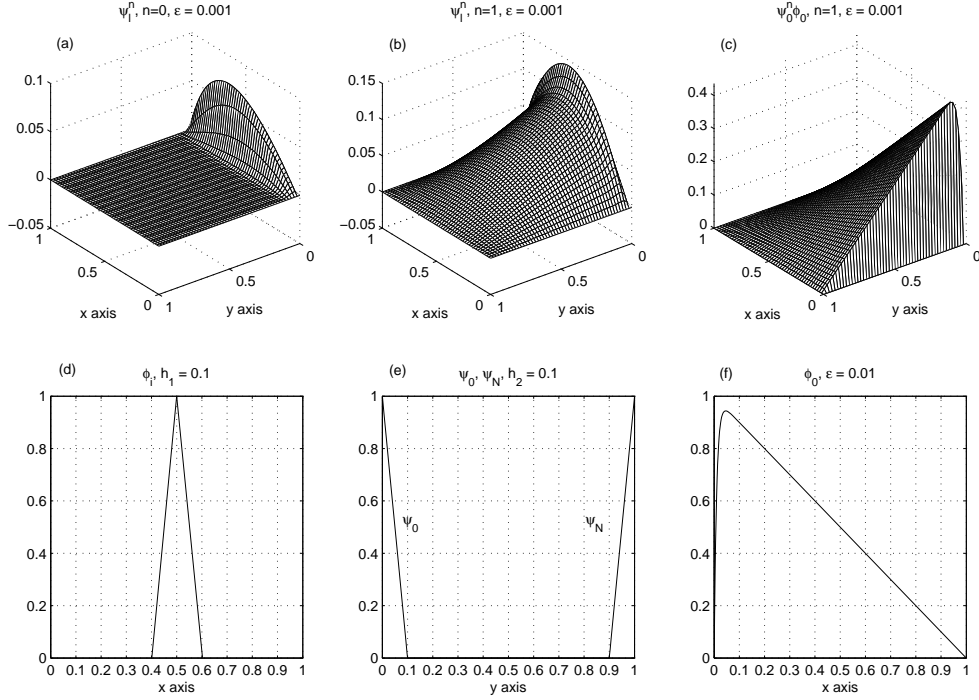


FIGURE 1. (a)(b) PBL elements  $\psi_l^n(x, y)$ ; (c) the overlapping of PBL and OBL  $\psi_0^n(y)\phi_0(x)$ ; (d)(e) bilinear elements,  $\phi_i(x)$ ,  $\psi_0(y)$ ,  $\psi_N(y)$ ; (f) OBL element,  $\phi_0(x)$ .

**3.2. Finite Element Spaces, Schemes, and Approximation Errors.** We now define the finite element spaces and consider the new schemes making use of the classical  $Q_1$  elements and the ordinary boundary layer element  $\phi_0$  as in (3.1) for  $V_N$  or adding to them the parabolic boundary layer elements as in (3.8)  $\psi_l^0, \dots, \psi_l^{n_0}$ ,

$\psi_u^0, \dots, \psi_u^{n_0}, \psi_0^0, \dots, \psi_0^{n_0}, \psi_N^0, \dots, \psi_N^{n_0}$  for  $\tilde{V}_N$ ; that is we introduce the spaces:

$$(3.24a) \quad V_N := \left\{ \sum_{j=1}^{N-1} c_{0j} \phi_0 \psi_j + \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} c_{ij} \phi_i \psi_j \right\} \subset H_0^1(\Omega),$$

$$(3.24b) \quad \tilde{V}_N := \left\{ \sum_{j=0}^{n_0} \alpha_j \psi_l^j + \sum_{j=0}^{n_0} \beta_j \psi_u^j + \sum_{j=0}^{n_0} \alpha_j \psi_0^j \phi_0 + \sum_{j=0}^{n_0} \beta_j \psi_N^j \phi_0 + v; v \in V_N \right\} \subset H_0^1(\Omega),$$

where  $\phi_0, \psi_l^j, \psi_u^j, \psi_0^j$ , and  $\psi_N^j$  are defined in (3.1), and (3.8); the constants  $\alpha_j$  and  $\beta_j$  are as in (3.4);  $\phi_i, \psi_j$ , for  $i = 1, \dots, M-1, j = 1, \dots, N-1$ , are bilinear elements w.r.t  $x$  and  $y$ , respectively, i.e., hat functions, see (d)(e) in Figure 1.

We now consider the three types of approximations corresponding to the three types of functions  $f$  in (3.3b)-(3.3d),  $f = f_1, f_2, f_3$  and the corresponding solutions  $u_1, u_2, u_3$  in (3.5). The general case follows by superposition. If  $f$  satisfies  $f(x, 0) = f(x, 1) = 0$  (i.e.  $f = f_1$ ), we look for an approximate solution  $u_N \in V_N$  such that

$$(3.25) \quad a_\epsilon(u_N, v) = \int_{\Omega} f_1 v d\Omega, \forall v \in V_N.$$

Thanks to (2.2), we derive the result of  $H^1$ - approximation error in Theorem 4.1 below quoted from [1]: note that we use the conditions (3.10) only, same as (2.60) as explained after (2.61).

**THEOREM 3.1.** *Assume that the conditions (3.10) on  $f = f(x, y)$  hold, namely,  $f = f_1$ . Let  $u = u_1^\epsilon$  be the exact solution of (1.3), and  $u_N$  the solution of (3.25). Then*

$$(3.26) \quad |u - u_N|_{H^1(\Omega)} \leq \kappa(\bar{h} + \bar{h}^2 \epsilon^{-1}).$$

If (3.10) is not satisfied,  $f \neq f_1$ , then parabolic boundary layers appear and we account for them by considering  $f_2$  and  $f_3$ .

We firstly notice that if  $f_2(x, 0) \neq 0$  or  $f_2(x, 1) \neq 0$ , then some coefficients  $\alpha_j$  or  $\beta_j$  are not zero and the parabolic boundary layers  $\varphi_l^0$  and  $\varphi_u^0$  corresponding to  $f_2$  and

$u_2$  appear as indicated from their explicit expression (2.18a). To handle them, we consider the approximate solution  $u_N^* \in \tilde{V}_N$  such that

$$(3.27a) \quad u_N^* = u_N + u^*,$$

where  $u_N$  is the solution of equation (3.25), and

$$(3.27b) \quad u^* = \sum_{j=0}^{n_0} \alpha_j \psi_l^j + \sum_{j=0}^{n_0} \beta_j \psi_u^j + \sum_{j=0}^{n_0} \alpha_j \psi_0^j \phi_0 + \sum_{j=0}^{n_0} \beta_j \psi_N^j \phi_0.$$

If  $f = f_2$ , then  $f_1 = f_3 = 0$ , and  $u_N = 0$ , we actually do not need to solve the linear system (3.25), and the approximate solution  $u_N^*$  can be found easily and explicitly from the data  $f$  as in (3.3c) and (3.4). We then expect that from Lemma 3.2,

$$(3.28) \quad |u - u_N^*|_{L^2} = |u - u^*|_{L^2} \leq \kappa \epsilon^{3/4};$$

notice also that the approximation errors due to the parabolic boundary layers do not affect the approximating system (3.25); the errors are totally independent of the discretization errors which arise in (3.25). To approximate the parabolic boundary layers  $\varphi_l^0$  and  $\varphi_u^0$ , a piecewise uniform mesh, which is only refined near  $y = 0, 1$  based on the pointwise estimate in Lemma 2.2, can be considered; but this will appear elsewhere. Here, as we mentioned before, we instead approximate the PBLs using the Lagrange interpolating polynomials as in (3.3c) and (3.4); they can be computed separately and independently of the discretized system (3.25).

Finally, to handle the term  $f_3$  corresponding to the truncating error of a Lagrange interpolation, we will need the following classical results on Lagrange interpolations, see e.g. [24], or Corollary 8.11 in [7].

**LEMMA 3.3.** *If  $P(x)$  is the Lagrange interpolating polynomial of degree at most  $n$  of a function  $g \in C^{n+1}([-1, 1])$  with nodes at the roots of the Chebyshev polynomial*

of degree  $n + 1$ , i.e.,

$$(3.29) \quad z_k = \cos \left( \frac{2k+1}{2(n+1)} \pi \right), \text{ for } k = 0, 1, \dots, n,$$

then

$$(3.30) \quad \max_{x \in [-1, 1]} |g(x) - P(x)| \leq \frac{1}{2^n(n+1)!} \max_{x \in [-1, 1]} |g^{(n+1)}(x)|.$$

By the change of variable  $\tilde{x} = (x+1)/2$ , we obtain the similar result on  $[0, 1]$ :

COROLLARY 3.1. *If  $P(x)$  is the Lagrange interpolating polynomial of degree at most  $n$  of  $g \in C^{n+1}([0, 1])$  with the nodes at*

$$(3.31) \quad z'_k = \frac{z_k + 1}{2}, \text{ for } k = 0, 1, \dots, n,$$

then

$$(3.32) \quad \max_{x \in [0, 1]} |g(x) - P(x)| \leq \frac{1}{2 \cdot 4^n(n+1)!} \max_{x \in [0, 1]} |g^{(n+1)}(x)|.$$

THEOREM 3.2. *For any  $f = f(x, y) \in C^\infty(\bar{\Omega})$ , let  $u = u^\epsilon$  be the exact solution of (1.3), and let  $u_N^*$  be defined as in (3.27). Then there exist positive constants  $\kappa$  independent of  $n_0$  and  $\epsilon$ , and  $\kappa(n_0)$  independent of  $\epsilon$  such that*

$$(3.33) \quad |u - u_N^*|_{H^1(\Omega)} \leq \kappa(\bar{h} + \bar{h}^2 \epsilon^{-1}) + \kappa(n_0) \epsilon^{1/4} + \frac{\kappa}{2 \cdot 4^{n_0}(n_0+1)! \epsilon^{1/2}} \max_{x \in [0, 1], y \in \{0, 1\}} |f^{(n_0+1)}(x, y)|.$$

PROOF. By the linearity of equation (1.3) and the uniqueness of solutions, we find

$$(3.34) \quad u = u_1 + u_2 + u_3 \in H_0^1(\Omega),$$

where  $u_j$ ,  $j = 1, 2, 3$  are as in (3.5) and (3.6). We have already obtained the approximation results for  $u_1$  and  $u_2$  in Theorem 4.1 and Lemma 3.2, respectively. We now majorize the norm  $|u_3|_{H^m}$ . From Lemma 2.7 applied to (3.6) with  $j = 3$ , we find that

$$(3.35) \quad \|u_3\|_\epsilon \leq \kappa |f_3|_{L^2}.$$

Having chosen the polynomials  $\sum_{j=0}^{n_0} \alpha_j x^j$  and  $\sum_{j=0}^{n_0} \beta_j x^j$  as the Lagrange interpolating polynomials for  $f(x, 0)$  and  $f(x, 1)$ , respectively, as in Corollary 3.1 or (3.3c) and (3.4), we find

$$\begin{aligned} |f_3(x, y)| &\leq (1-y) \left| f(x, 0) - \sum_{j=0}^{n_0} \alpha_j x^j \right| + y \left| f(x, 1) - \sum_{j=0}^{n_0} \beta_j x^j \right| \\ &\leq \frac{1}{2 \cdot 4^{n_0} (n_0 + 1)!} \left\{ (1-y) \max_{x \in [0,1]} |f^{(n_0+1)}(x, 0)| + y \max_{x \in [0,1]} |f^{(n_0+1)}(x, 1)| \right\} \\ &\leq \frac{1}{2 \cdot 4^{n_0} (n_0 + 1)!} \max_{x \in [0,1], y \in \{0,1\}} |f^{(n_0+1)}(x, y)|. \end{aligned}$$

Writing

$$|u - u_N^*|_{H^1} \leq |u_1 + u_2 + u_3 - u_N - u^*|_{H^1} \leq |u_1 - u_N|_{H^1} + |u_2 - u^*|_{H^1} + |u_3|_{H^1},$$

the theorem now follows from Theorem 4.1, and from Lemma 3.2.  $\square$

**REMARK 3.2.** From Theorem 4.1 and Theorem 4.2, we find that for the schemes (3.25) and (3.27) to be effective, we require the space mesh to be of order  $\bar{h} = o(\epsilon^{1/2})$  in the  $H^1$  approximation. The  $L^2$ - error estimate can be derived via an  $L^2$ - stability analysis which will appear in Chapter 3.

#### 4. A Mixed Boundary Value Problem

We now consider another type of boundary conditions for which the effects of the parabolic boundary layers at  $y = 0, 1$  are milder. We consider the mixed boundary value problem (1.1a),(1.1c). Its weak formulation is as in (1.3),  $H_0^1(\Omega)$  being replaced by

$$(4.1) \quad V = \left\{ v \in H^1(\Omega); v = 0 \text{ at } x = 0, 1 \right\}.$$

The error estimates for the approximate solutions and numerical simulations of the mixed boundary value problem are shown in [1] under strong conditions on  $f$ , namely,  $f_y, f_{yyy} = 0$  at  $y = 0, 1$ . These conditions make the normal derivatives of  $u^0$  and  $u^1$ ,



which are obtained from the explicit solutions in (2.4), vanish at  $y = 0, 1$ . This suppresses the occurrences of parabolic boundary layers, see [1]. But in general, we do not expect that the normal derivatives of  $u^j$  vanish at  $y = 0, 1$ . When this happens, to remove the discrepancies with the second condition (1.1c) at  $y = 0$ , we consider the following parabolic equations for  $\bar{\varphi}_l^j$ :

$$(4.2a) \quad \begin{aligned} O(1) : & -\bar{\varphi}_{l\bar{y}\bar{y}}^0 - \bar{\varphi}_{lx}^0 = 0, \\ O(\epsilon^j) : & -\bar{\varphi}_{l\bar{y}\bar{y}}^j - \bar{\varphi}_{lx}^j = \bar{\varphi}_{lxx}^{j-1}, \text{ for } j \geq 1. \end{aligned}$$

The boundary conditions are

$$(4.2b) \quad \bar{\varphi}_l^j(x, \bar{y}) = 0, \text{ at } x = 1,$$

$$(4.2c) \quad \frac{\partial \bar{\varphi}_l^j}{\partial \bar{y}}(x, \bar{y} = 0) = \epsilon^{1/2} R_j(x)^3,$$

$$(4.2d) \quad \bar{\varphi}_l^j(x, \bar{y}) \rightarrow 0 \text{ as } \bar{y} \rightarrow \infty,$$

where  $R_j(x) = -\partial u^j / \partial y(x, 0)$ .

The explicit form of the  $\bar{\varphi}_l^j$ , and its pointwise and norm estimates are provided by the following lemma. As before, we consider a heat equation in a semi-strip, but this time with a flux boundary condition, see Theorem 20.3.2 in [8]. Let

$$(4.3) \quad D = \{(x, y) \in \mathbb{R}^2; 0 < x < 1, y > 0\}.$$

We are given  $f^*$  which is uniformly Hölder continuous in  $x$  and  $y$  for each compact subset of  $D$  and satisfies

$$(4.4) \quad |f^*(x, y)| \leq \kappa \epsilon^{1/2} \exp(-\gamma y),$$

---

<sup>3</sup>The boundary condition (4.2c) is equivalent to:  $\partial \bar{\varphi}_l^j / \partial y(x, y = 0) = R_j(x)$ , and then we easily see that  $\partial \bar{\varphi}_l^j / \partial y$  resolves the discrepancies of  $-\partial u^j / \partial y$  at  $y = 0$ .

for some  $\gamma > 0$ , and all  $0 < x < 1$  and  $y > 0$ ; we are also given  $g^*$  which is continuous on  $[0, 1]$ . We look for  $u$  satisfying:

$$(4.5) \quad \begin{cases} -\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial y^2} = \epsilon^{1/2} f^*, \text{ for } (x, y) \in D, \\ \frac{\partial u}{\partial y}(x, 0) = \epsilon^{1/2} g^*(x), \text{ } 0 < x < 1, \\ u(x, y) \rightarrow 0 \text{ as } y \rightarrow \infty, \text{ } 0 < x < 1, \\ u(1, y) = 0. \end{cases}$$

**Compatibility Conditions.** We consider as before the following smoothness and compatibility conditions on the data  $f^*$ ,  $g^*$  to attain  $u \in C^l(\bar{D})$ ,  $l \geq 0$ :

$$(4.6a) \quad f^*(x, y) \text{ and } g^*(x) \text{ are sufficiently smooth on } \bar{D} \text{ and } [0, 1], \text{ respectively,}$$

and

$$(4.6b) \quad \frac{\partial^i}{\partial x^i} f^*(1, y) = \frac{\partial^i}{\partial x^i} g^*(1) = 0, \text{ for } 0 \leq i \leq l.$$

Differentiating (2.10) in  $y$  with  $f^*$  being replaced by  $\epsilon^{1/2} f^*$ , and setting  $x = 1$ , from the boundary conditions of (4.5), we then find

$$(4.7) \quad (-1)^k \frac{\partial^k}{\partial x^k} g^*(1) + \sum_{s=0}^{k-1} (-1)^{s+1} \frac{\partial^{2k-s-1}}{\partial x^s \partial y^{2(k-s-1)+1}} f^*(1, y) = 0, \text{ } k = 0, 1, \dots, l,$$

which is necessary for  $u \in C^l(\bar{D})$ ; conditions (4.6) are much stronger than (4.7).

From now on we thus assume that: for  $0 \leq i \leq 2n+d-2j$ ,  $d = 0, 1$ , and  $0 \leq j \leq n$ ,

$$(4.8) \quad R_j^{(i)}(1) = -\frac{\partial^{i+1}}{\partial x^i \partial y} u^j(1, 0) = 0.$$

We then find results similar to Lemma 2.1, that is:

LEMMA 4.1. *Let  $u = u(x, y)$  be the solution of the heat equation (4.5) in  $D$ : Then the solution  $u$  is unique and it admits the integral representation:*

$$(4.9) \quad \begin{aligned} u(x, y) = & -\frac{\epsilon^{1/2}}{\sqrt{\pi}} \int_0^{1-x} \frac{1}{\sqrt{t}} \exp\left(-\frac{y^2}{4t}\right) g^*(x+t) dt \\ & + \frac{\epsilon^{1/2}}{2\sqrt{\pi}} \int_0^{1-x} \int_0^\infty \frac{1}{\sqrt{s}} \left\{ \exp\left[-\frac{(y-t)^2}{4s}\right] + \exp\left[-\frac{(y+t)^2}{4s}\right] \right\} f^*(x+s, t) dt ds, \end{aligned}$$

and

$$(4.10) \quad |u(x, y)| \leq \kappa \exp(-\gamma y), \text{ for the same } \gamma \text{ as in (4.4).}$$

If the conditions (4.6) hold, then  $u \in C^l(\bar{D})$ ,  $l \geq 0$ . Furthermore, if the following decay conditions hold:

$$(4.11) \quad \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} f^*(x, y) \right| \leq \kappa \epsilon^{1/2} \exp(-\gamma y), \text{ for } 0 \leq i+m \leq l+1, \text{ some } \gamma > 0,$$

then the following pointwise estimates for  $u$  and its derivatives hold: for each  $i$  and  $m$ , there exists a constant  $\kappa_{im}$  which depends only on  $f^*$  and  $g^*$  such that

$$(4.12) \quad \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} u(x, y) \right| \leq \kappa_{im} \epsilon^{1/2} \exp(-\gamma y), \forall (x, y) \in D,$$

for  $0 \leq i+m \leq l+1$ , the same  $\gamma$  as in (4.11).

For the proof, see the Appendix.

REMARK 4.1. From Lemma 4.1, if (4.6) and (4.11) hold, it is obvious that, as before, the regularity properties (2.17) hold.  $\square$

We find the solutions  $\bar{\varphi}_l^j$  of equation (4.2) recursively:

(4.13a)

$$\bar{\varphi}_l^0(x, \bar{y}) = -\frac{\epsilon^{1/2}}{\sqrt{\pi}} \int_0^{1-x} \frac{1}{\sqrt{t}} \exp\left(-\frac{y^2}{4t}\right) R_0(x+t) dt.$$

(4.13b)

$$\begin{aligned} \bar{\varphi}_l^j(x, \bar{y}) = & -\frac{\epsilon^{1/2}}{\sqrt{\pi}} \int_0^{1-x} \frac{1}{\sqrt{t}} \exp\left(-\frac{y^2}{4t}\right) R_j(x+t) dt \\ & + \frac{\epsilon^{1/2}}{2\sqrt{\pi}} \int_0^{1-x} \int_0^\infty \frac{1}{\sqrt{s}} \left\{ \exp\left[-\frac{(\bar{y}-t)^2}{4s}\right] + \exp\left[-\frac{(\bar{y}+t)^2}{4s}\right] \right\} \frac{\partial^2}{\partial x^2} \bar{\varphi}_l^{j-1}(x+s, t) dt ds, \end{aligned}$$

for  $1 \leq j \leq n$ .

Notice that the  $\bar{\varphi}_l^j$  resolve the discrepancies of the normal derivatives of  $u^j$  at  $y = 0$ . Furthermore, thanks to the compatibility conditions (4.8), we find that for  $0 \leq j \leq n$ ,  $\bar{\varphi}_l^j(x, \bar{y}) = \bar{\varphi}_l^j(x, y/\sqrt{\epsilon})$  satisfies the regularities (2.17) with  $l = 2n + d - 2j$ , and  $y = \bar{y}$ .

The following lemma can be deduced from (4.12) directly and this lemma provides the derivative estimates for  $\bar{\varphi}_l^j$  which will be used for asymptotic error estimates later on. Furthermore, as indicated in the pointwise and norm estimates in the two subsequent lemmas, it turns out that these boundary layers are not crucial, i.e. they are mild, unlike the parabolic boundary layers in the Dirichlet boundary value problem (1.1a),(1.1b).

**LEMMA 4.2.** *Assume that the conditions (4.8) hold. Then there exist a positive constants  $\kappa$  independent of  $\epsilon$  such that the following inequalities hold*

$$(4.14) \quad \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \bar{\varphi}_l^j \left( x, \frac{y}{\sqrt{\epsilon}} \right) \right| \leq \kappa_{ijm} \epsilon^{-m/2+1/2} \exp\left(-\frac{y}{\sqrt{\epsilon}}\right), \forall (x, y) \in \bar{\Omega},$$

for  $0 \leq i + m \leq 2n + d + 1 - 2j$ , and  $0 \leq j \leq n$ .

The following norm estimates are deduced from Lemma 4.2.

LEMMA 4.3. *Assume that the conditions (4.8) hold. Let, for  $0 \leq \sigma < 1$ ,*

$$\Omega^\sigma = (0, 1) \times (\sigma, 1).$$

*Then there exist positive constants  $\kappa$  and  $c$  independent of  $\epsilon$  such that the following inequalities hold: for  $0 \leq i + m \leq 2n + d + 1 - 2j$ ,*

$$(4.15a) \quad \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \bar{\varphi}_l^j \right|_{L^2(\Omega^\sigma)} \leq \kappa_{ijm} \epsilon^{-m/2+3/4} \exp\left(-\frac{\sigma}{\sqrt{\epsilon}}\right);$$

*in particular,*

$$(4.15b) \quad \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \bar{\varphi}_l^j \right|_{L^2(\Omega)} \leq \kappa_{ijm} \epsilon^{-m/2+3/4}.$$

REMARK 4.2. Similarly, at  $y = 1$ , we may introduce  $\bar{\varphi}_u^j$  which have the same structure as  $\bar{\varphi}_l^j$ . Then similarly to (4.8), we will need the following conditions:

$$(4.16) \quad -\frac{\partial^{i+1}}{\partial x^i \partial y} u^j(1, 1) = 0, \text{ for } 0 \leq i \leq 2n + d - 2j, d = 0, 1, \text{ and } 0 \leq j \leq n.$$

We also notice that  $\bar{\varphi}_u^j(x, \tilde{y}) = \bar{\varphi}_u^j(x, (1 - y)/\sqrt{\epsilon})$  satisfies the regularities (2.17) with  $l = 2n + d - 2j$ , and  $y = \tilde{y}$ . Similarly Lemma 4.2 and Lemma 4.3 are valid with  $\bar{\varphi}_l^j$  replaced by  $\bar{\varphi}_u^j$ ,  $\bar{y}$  by  $\tilde{y}$ , and  $(\sigma, 1)$  by  $(0, 1 - \sigma)$ .  $\square$

We now have to resolve the discrepancies at  $x = 0$  due to  $u^j$ ,  $\bar{\varphi}_l^j$ , and  $\bar{\varphi}_u^j$ ; we thus define  $\bar{\theta}^j = \bar{\theta}^j(\bar{x}, y)$  as  $\theta^j$  before, and we can derive the pointwise and norm estimates in the following lemmas as in Lemma 2.5 and Lemma 2.6; the proof is similar. The explicit solutions can be found as before: for  $j = 0, 1$ ,

$$(4.17) \quad \begin{aligned} \bar{\theta}^j\left(\frac{x}{\epsilon}, y\right) = & -\left(u^j(0, y) + \bar{\varphi}_l^j\left(0, \frac{y}{\sqrt{\epsilon}}\right) + \bar{\varphi}_u^j\left(0, \frac{1-y}{\sqrt{\epsilon}}\right)\right) \exp\left(-\frac{x}{\epsilon}\right) \\ & + e.s.t.(2n + d + 1 - 2j). \end{aligned}$$

LEMMA 4.4. *Assume that the conditions (4.8) and (4.16) hold. For any  $0 < c < 1$ , there exist a positive constant  $\kappa_{ijm}$  independent of  $\epsilon$  such that the following inequalities hold: for  $0 \leq i + m \leq 2n + d + 1 - 2j$ , and  $j = 2k$  or  $j = 2k + 1$  with  $k \geq 0$ , for all*

$(x, y) \in \bar{\Omega}$ ,

$$(4.18) \quad \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \bar{\theta}^j \left( \frac{x}{\epsilon}, y \right) \right| \leq \kappa_{ijm} \epsilon^{-i} \exp \left( -c \frac{x}{\epsilon} \right) \left\{ 1 + \epsilon^{-k-m/2+1/2} \exp \left( -\frac{y}{\sqrt{\epsilon}} \right) \right. \\ \left. + \epsilon^{-k-m/2+1/2} \exp \left( -\frac{1-y}{\sqrt{\epsilon}} \right) \right\} + e.s.t. (2n + d + 1 - 2j).$$

LEMMA 4.5. For  $0 \leq \sigma_1, \sigma_2 < 1$ , let

$$\Omega^{\sigma_1, \sigma_2} = (\sigma_1, 1) \times (\sigma_2, 1 - \sigma_2).$$

Assume that the conditions (4.8) and (4.16) hold. For any  $0 < c < 1$ , there exist a positive constant  $\kappa_{ijm}$  independent of  $\epsilon$  such that the following inequalities hold, for  $0 \leq i \leq 2n + d + 1 - 2j$ , and  $j = 2k$  or  $j = 2k + 1$  with  $k \geq 0$ ,

$$(4.19a) \quad \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \bar{\theta}^j \right|_{L^2(\Omega^{\sigma_1, \sigma_2})} \leq \kappa_{ijm} \epsilon^{-i+1/2} \\ \cdot \left( 1 + \epsilon^{-k-m/2+3/4} \exp \left( -\frac{\sigma_2}{\sqrt{\epsilon}} \right) \right) \exp \left( -c \frac{\sigma_1}{\epsilon} \right);$$

in particular,

$$(4.19b) \quad \left| \frac{\partial^{i+m}}{\partial x^i \partial y^m} \bar{\theta}^j \right|_{L^2(\Omega)} \leq \kappa_{ijm} \epsilon^{-i+1/2} (1 + \epsilon^{-k-m/2+3/4}).$$

We now consider the following compatibility conditions:

$$(4.20) \quad \frac{\partial}{\partial y} f(1, 0) = \frac{\partial}{\partial y} f(1, 1) = 0;$$

note that the compatibility conditions are much weaker than in [1].

LEMMA 4.6. Assume that (4.20) hold. Then

$$(4.21a) \quad |u^\epsilon - u^0 - \bar{\varphi}_l^0 - \bar{\varphi}_u^0 - \bar{\theta}^0 - \bar{\delta}^\epsilon|_{L^2(\Omega)} \leq \kappa \epsilon^{7/4},$$

$$(4.21b) \quad \|u^\epsilon - u^0 - \bar{\varphi}_l^0 - \bar{\varphi}_u^0 - \bar{\theta}^0 - \bar{\delta}^\epsilon\|_{H^1(\Omega)} \leq \kappa \epsilon^{5/4},$$

$$(4.21c) \quad \|u^\epsilon - u^0 - \bar{\varphi}_l^0 - \bar{\varphi}_u^0 - \bar{\theta}^0 - \bar{\delta}^\epsilon\|_{H^2(\Omega)} \leq \kappa \epsilon^{1/4},$$

where the function  $\bar{\delta}^\epsilon \in V$  is described in the proof and such that:

$$(4.22) \quad |\bar{\delta}^\epsilon|_{L^2(\Omega)} \leq \kappa \epsilon, \quad \|\bar{\delta}^\epsilon\|_{H^1(\Omega)} \leq \kappa \epsilon^{1/2}.$$

PROOF. For  $n = 0$ , from the conditions (4.20), we can easily verify that

$$-\frac{\partial^{i+1}}{\partial x^i \partial y} u^0(1, 0) = -\frac{\partial^{i+1}}{\partial x^i \partial y} u^0(1, 1) = 0, \text{ for } 0 \leq i \leq 1;$$

we find that  $n = 0$ ,  $d = 1$  in (4.8) and (4.16) and hence,  $\bar{\varphi}_l^0, \bar{\varphi}_u^0, \bar{\theta}^0 \in H^2(\Omega)$ . Let

$$(4.23) \quad \bar{w}_{\epsilon 0} = u^\epsilon - u^0 - \bar{\varphi}_l^0 - \bar{\varphi}_u^0 - \bar{\theta}^0.$$

Then similarly to (2.41) and (2.46), setting

$$(4.24) \quad \bar{v}^0 = -\bar{w}_{\epsilon 0}(x, 0) \frac{(y-1)^2}{2} + \bar{w}_{\epsilon 0}(x, 1) \frac{y^2}{2},$$

we find that  $\bar{w}_{\epsilon 0} - \bar{v}^0$  satisfies the boundary condition (1.1c) on  $\partial\Omega$ . Since from Lemma 4.2, Remark 4.2, and Lemma 4.4, we easily find that, similarly to (2.42),  $\bar{v}^0$  is exponentially small and it is absorbed in other norms; we may drop the  $\bar{v}^0$ . Hence, we write

$$(4.25a) \quad L_\epsilon \bar{w}_{\epsilon 0} = \bar{R}_1 + \bar{R}_2,$$

$$(4.25b) \quad \bar{w}_{\epsilon 0} = 0 \text{ at } x = 0, 1,$$

$$(4.25c) \quad \frac{\partial \bar{w}_{\epsilon 0}}{\partial y} = 0 \text{ at } y = 0, 1,$$

where

$$(4.26a) \quad \bar{R}_1 = \epsilon \{ \bar{\varphi}_{lxx}^0 + \bar{\varphi}_{uxx}^0 \},$$

$$(4.26b) \quad \bar{R}_2 = \epsilon \{ \triangle u^0 + \bar{\theta}_{yy}^0 \}.$$

Let  $\bar{\delta}$  be the solution of:

$$(4.27a) \quad L_\epsilon \bar{\delta}^\epsilon = \bar{R}_2 \text{ in } \Omega,$$

$$(4.27b) \quad \bar{\delta}^\epsilon = 0 \text{ at } x = 0, 1,$$

$$(4.27c) \quad \frac{\partial \bar{\delta}^\epsilon}{\partial y} = 0 \text{ at } y = 0, 1.$$

Then we easily find that  $|\bar{R}_2|_{L^2} \leq \kappa\epsilon$ , and hence, using Lemma 2.7, the estimate (4.22) follows. Furthermore, we find

$$(4.28a) \quad L_\epsilon(\bar{w}_{\epsilon 0} - \bar{\delta}^\epsilon) = \bar{R}_1 \text{ in } \Omega,$$

$$(4.28b) \quad \bar{w}_{\epsilon 0} - \bar{\delta}^\epsilon = 0 \text{ at } x = 0, 1,$$

$$(4.28c) \quad \frac{\partial}{\partial y}(\bar{w}_{\epsilon 0} - \bar{\delta}^\epsilon) = 0 \text{ at } y = 0, 1.$$

Since  $|\bar{R}_1|_{L^2} \leq \kappa\epsilon^{7/4}$ , by applying Lemma 2.7 to  $\bar{w}_{\epsilon 0} - \bar{\delta}^\epsilon$  again, the lemma follows.  $\square$

REMARK 4.3. Assume that (4.20) holds. Then from Lemma 4.6 using the norm estimates of Lemma 4.3 and Lemma 4.5, we obtain

$$u^\epsilon = u^0 + O(\epsilon^{1/2}) \text{ in } L^2. \quad \square$$

In the following lemma we will see that the boundary layer element  $\phi_0$  introduced in (3.1) absorbs the  $H^2$ - singularity of  $u^\epsilon$ .

LEMMA 4.7. *Assume that (4.20) hold. Then there exist a positive constant  $\kappa$  independent of  $\epsilon$ , and a smooth function  $\bar{g} = \bar{g}^\epsilon(y)$  with  $|\bar{g}|_{H^2(0,1)} \leq \kappa\epsilon^{-1/4}$  such that*

$$(4.29) \quad \left\| u^\epsilon - \bar{g}\phi_0 - \tilde{\delta}^\epsilon - \bar{\delta}^\epsilon \right\|_{H^2(\Omega)} \leq \kappa,$$

where  $\bar{\delta}^\epsilon \in V$  is as in Lemma 4.6, and the function  $\tilde{\delta}^\epsilon \in V$  and its derivatives are estimated as follows:

$$(4.30) \quad \left| \frac{\partial^{i+m} \tilde{\delta}^\epsilon}{\partial x^i \partial y^m} \right|_{L^2(\Omega)} \leq \kappa \begin{cases} \epsilon^{3/4} & \text{for } m = 0, i = 0, 1, 2, \\ \epsilon^{1/4} & \text{for } m = 1, i = 0, 1, \\ \epsilon^{-1/4} & \text{for } m = 2, i = 0. \end{cases}$$

PROOF. From the asymptotic error (4.21c), we find

$$(4.31) \quad \|u^\epsilon - \bar{\varphi}_l^0 - \bar{\varphi}_u^0 - \bar{\theta}^0 - \bar{\delta}^\epsilon\|_{H^2} \leq \kappa.$$



Notice that from the condition (4.20),  $\bar{\varphi}_l^0, \bar{\varphi}_u^0, \bar{\theta}^0 \in H^2(\Omega)$ , see the proof in Lemma 4.6. Then from the explicit solution  $\bar{\theta}^0$  in (4.17), with the definition  $\phi_0$  in (3.1), we find

$$(4.32a) \quad \bar{\varphi}_l^0 + \bar{\varphi}_u^0 + \bar{\theta}^0 = \bar{g}\phi_0(x) + \tilde{\delta}^\epsilon + u^0(0, y)(x - 1) + e.s.t.(2),$$

where

$$(4.32b) \quad \bar{g} = \bar{g}^\epsilon(y) = u^0(0, y) + \bar{\varphi}_l^0(0, \bar{y}) + \bar{\varphi}_u^0(0, \tilde{y}),$$

$$(4.32c) \quad \tilde{\delta}^\epsilon = (\bar{\varphi}_l^0(0, \bar{y}) + \bar{\varphi}_u^0(0, \tilde{y}))(x - 1) + \bar{\varphi}_l^0(x, y) + \bar{\varphi}_u^0(x, y) \in V;$$

note that from the boundary condition  $\bar{\varphi}_l^0 = \bar{\varphi}_u^0 = 0$  at  $x = 1$ . The estimate (4.30) and the estimate for  $|g|_{H^2}$  follow from Lemma 4.2 - 4.3, and hence the lemma follows.  $\square$

To approximate the solutions of (4.1), we introduce the following finite element space below.

$$(4.33) \quad \bar{V}_N := \left\{ \sum_{j=0}^N c_{0j} \phi_0 \psi_j + \sum_{i=1}^{M-1} \sum_{j=0}^N c_{ij} \phi_i \psi_j \right\} \subset V,$$

where  $\phi_0$  is defined in (3.1),  $\phi_i, \psi_j$  are piecewise bilinear elements w.r.t  $x$  and  $y$ , respectively, on a *uniform mesh* for  $i = 1 \cdots M - 1, j = 0, 1 \cdots N - 1, N$  as in (d)(e) in Figure 1, and  $V$  is defined in (4.1). We look for an approximate solution  $u_N \in \bar{V}_N$  such that

$$(4.34) \quad a_\epsilon(u_N, v) = F(v), \forall v \in \bar{V}_N.$$

Notice that the approximating system (4.34) has ordinary boundary layer element  $\phi_0$ , and two hat functions  $\psi_0, \psi_N$  as in (e) in Figure 1.

As before, we will need the following interpolation inequalities to derive the approximation errors.

LEMMA 4.8. *Assume that (4.20) hold. Then there exists an interpolant  $\tilde{u}_N \in \bar{V}_N$  such that*

$$(4.35) \quad \|u^\epsilon - \tilde{u}_N - \bar{\delta}^\epsilon\|_{L^2(\Omega)} \leq \kappa(h_1^2 + h_2^2\epsilon^{-1/4}),$$

$$(4.36) \quad \|u^\epsilon - \tilde{u}_N - \bar{\delta}^\epsilon\|_{H^1(\Omega)} \leq \kappa(h_1 + h_2^2\epsilon^{-3/4} + h_2\epsilon^{-1/4}).$$

PROOF. By the classical interpolation results, see e.g., [3], [11], [26], applied to  $u^\epsilon - \bar{g}\phi_0 - \tilde{\delta}^\epsilon - \bar{\delta}^\epsilon \in V$  using Lemma 4.7, there exist the  $c_{ij}$  such that for  $m = 0, 1$ ,

$$(4.37) \quad \begin{aligned} I_1(m) &:= \left\| u^\epsilon - \bar{g}\phi_0 - \tilde{\delta}^\epsilon - \bar{\delta}^\epsilon - \sum_{i=1}^{M-1} \sum_{j=0}^N c_{ij} \phi_i \psi_j \right\|_{H^m(\Omega)} \\ &\leq \kappa \bar{h}^{2-m} |u^\epsilon - \bar{g}\phi_0|_{H^2(\Omega)} \leq \kappa \bar{h}^{2-m}. \end{aligned}$$

Then using Lemma 4.7 again, we derive the following estimates. There exist the  $c_{0j}$  such that for  $m = 0, 1$ ,

$$(4.38) \quad \left\| \bar{g} - \sum_{j=0}^N c_{0j} \psi_j \right\|_{H^m(\Omega)} \leq \kappa h_2^{2-m} |\bar{g}|_{H^2(0,1)} \leq \kappa h_2^{2-m} \epsilon^{-1/4}.$$

Hence, we find that for  $m = 0, 1$ ,

$$(4.39) \quad \begin{aligned} I_2(m) &:= \left| \bar{g}\phi_0 - \sum_{j=0}^N c_{0j} \phi_0 \psi_j \right|_{H^m(\Omega)} \\ &\leq \kappa \begin{cases} \left| \bar{g} - \sum_{j=0}^N c_{0j} \psi_j \right|_{L^2(0,1)} \\ \epsilon^{-1/2} \left| \bar{g} - \sum_{j=0}^N c_{0j} \psi_j \right|_{L^2(0,1)} + \left| \bar{g} - \sum_{j=0}^N c_{0j} \psi_j \right|_{H^1(0,1)} \end{cases} \\ &\leq \kappa \begin{cases} h_2^2 \epsilon^{-1/4} & \text{for } m = 0, \\ h_2^2 \epsilon^{-3/4} + h_2 \epsilon^{-1/4} & \text{for } m = 1. \end{cases} \end{aligned}$$

By the classical interpolation results, see e.g., [26], or Lemma 3.3 in [3], applied to  $\tilde{\delta} = \bar{\delta}^\epsilon \in V$  with the estimates (4.30), we find that there exist the  $c_{ij}$  such that for

$m = 0, 1,$

$$\begin{aligned}
 (4.40) \quad I_3(m) &:= \left\| \tilde{\delta} - \sum_{i=1}^{M-1} \sum_{j=0}^N c_{ij} \phi_i \psi_j \right\|_{L^2(\Omega)} \\
 &\leq \kappa \begin{cases} h_1^2 |\partial^2 \tilde{\delta} / \partial x^2|_{L^2} + h_2^2 |\partial^2 \tilde{\delta} / \partial y^2|_{L^2} + h_1 h_2 |\partial^2 \tilde{\delta} / \partial x \partial y|_{L^2} \\ h_1 |\partial^2 \tilde{\delta} / \partial x^2|_{L^2} + h_2 |\partial^2 \tilde{\delta} / \partial y^2|_{L^2} + \bar{h} |\partial^2 \tilde{\delta} / \partial x \partial y|_{L^2} \end{cases} \\
 &\leq \kappa \begin{cases} h_1^2 \epsilon^{3/4} + h_2^2 \epsilon^{-1/4} + h_1 h_2 \epsilon^{1/4} & \text{for } m = 0, \\ h_1 \epsilon^{3/4} + h_2 \epsilon^{-1/4} + \bar{h} \epsilon^{1/4} & \text{for } m = 1. \end{cases}
 \end{aligned}$$

Hence, we easily verify that

$$(4.41) \quad \left\| u^\epsilon - \sum_{j=0}^N c_{0j} \phi_0 \psi_j - \sum_{i=1}^{M-1} \sum_{j=0}^N c_{ij} \phi_i \psi_j - \bar{\delta}^\epsilon \right\|_{H^m} \leq I_1(m) + I_2(m) + I_3(m),$$

and setting

$$(4.42) \quad \tilde{u}_N = \sum_{j=0}^N c_{0j} \phi_0 \psi_j + \sum_{i=1}^{M-1} \sum_{j=0}^N c_{ij} \phi_i \psi_j,$$

the lemma follows.  $\square$

We are now able to derive the error estimates below using Lemma 4.8. But by the subtlety of the a priori estimates involving the small parameter  $\epsilon$ , the term  $\bar{\delta}^\epsilon$  should be dealt with caution, see (4.45) below.

**THEOREM 4.1.** *Assume that the conditions (4.20) on  $f = f(x, y)$  hold. Let  $u_N$  be the solution of (4.34) and  $\tilde{u}_N$  the interpolant in  $\bar{V}_N$  defined in Lemma 4.8. Then*

$$\begin{aligned}
 (4.43) \quad |u_N - \tilde{u}_N|_{H^1(\Omega)} &\leq \kappa (h_1 + h_1^2 \epsilon^{-1} + h_2 \epsilon^{-1/4} + h_2^2 \epsilon^{-5/4} + \epsilon^{1/2}) \\
 &\quad + \kappa \epsilon^{-1/2} |u_N - \tilde{u}_N|_{L^2(\Omega)}.
 \end{aligned}$$

**PROOF.** Subtracting (4.34) from (4.1), we find

$$(4.44) \quad a_\epsilon(u - u_N, v) = 0, \quad \forall v \in \bar{V}_N,$$

and hence

$$\begin{aligned}
 a_\epsilon(u_N - \tilde{u}_N, u_N - \tilde{u}_N) &= a_\epsilon(u - \tilde{u}_N, u_N - \tilde{u}_N) \\
 (4.45) \quad &= (\text{from equation (4.27) for } \bar{\delta}^\epsilon) \\
 &= a_\epsilon(u - \tilde{u}_N - \bar{\delta}, u_N - \tilde{u}_N) + (\bar{R}_2, u_N - \tilde{u}_N).
 \end{aligned}$$

Then we find

$$\begin{aligned}
 \epsilon |\nabla(u_N - \tilde{u}_N)|_{L^2}^2 &\leq \kappa \epsilon |\nabla(u - \tilde{u}_N - \bar{\delta})|_{L^2}^2 + \kappa \epsilon^{-1} |u - \tilde{u}_N - \bar{\delta}|_{L^2}^2 \\
 (4.46) \quad &+ \kappa |\bar{R}_2|_{L^2}^2 + \kappa |u_N - \tilde{u}_N|_{L^2}^2.
 \end{aligned}$$

Since  $|\bar{R}_2|_{L^2} \leq \kappa \epsilon$ , we easily verify that

$$\begin{aligned}
 |u_N - \tilde{u}_N|_{H^1} &\leq \kappa |u - \tilde{u}_N - \bar{\delta}|_{H^1} + \kappa \epsilon^{-1} |u - \tilde{u}_N - \bar{\delta}|_{L^2} \\
 (4.47) \quad &+ \kappa \epsilon^{1/2} + \kappa \epsilon^{-1/2} |u_N - \tilde{u}_N|_{L^2};
 \end{aligned}$$

the theorem follows from the interpolation inequalities in Lemma 4.8.  $\square$

REMARK 4.4. The  $L^2$ -error,  $|u_N - \tilde{u}_N|_{L^2(\Omega)}$ , will be derived using the  $L^2$ -stability analysis which will appear in Chapter 3. Then the  $H^1$ -error,  $|u_N - \tilde{u}_N|_{H^1(\Omega)}$ , will be easily found from Theorem 4.1; thus  $|u^\epsilon - \tilde{u}_N|_{H^m(\Omega)}$ ,  $m = 0, 1$ , will follow.

## 5. Occurrence of Boundary Layers

In this section, we summarize the type of the occurrences of boundary layers using the model equation:

$$(5.1) \quad -\epsilon \Delta u - u_x = f \quad \text{in } \Omega = (0, 1) \times (0, 1);$$

the boundary conditions are specified in the table below. More general singularly perturbed equations will appear elsewhere. But this simple model equation covers two major boundary layers which essentially affect numerical computations in general cases.

From the lower-order asymptotic analysis, i.e. with  $n = 0, 1$ , as we did in previous sections, we are able to detect two major boundary layers which are ordinary

and parabolic boundary layers at the outflow and at the characteristic boundaries, respectively. We notice that these are determined from the data  $f$  and the boundary conditions. More precisely, for the Dirichlet boundary value problem (1.1a), (1.1b), if

$$(5.2) \quad f = 0 \quad \text{at } y = 0, 1,$$

parabolic boundary layers are suppressed. In this case, we only observe ordinary boundary layers at the outflow (i.e. at  $x = 0$  in problem (5.1)). Then the discretization errors due to them are propagated in the whole domain due to the convective term (i.e. in the  $x$ -direction due to the term  $u_x$  in problem (5.1)). If the discretization errors are not properly handled, the approximate solutions display wild oscillations in the propagation direction, see Figure 2 below.

Another way to suppress parabolic boundary layers is imposing the boundary conditions as in the mixed boundary value problem (1.1a), (1.1c), see Section 4; obviously, we can expect that from condition (1.1c) (the normal derivatives of the exact solution  $u^\epsilon$  vanish at the characteristic boundaries), the  $u^\epsilon$  varies slowly at those characteristic boundaries and thus the parabolic boundary layers are mild.

For the case where parabolic boundary layers stand out, since the discretization errors due to them are localized near the characteristic boundaries, i.e.  $y = 0, 1$ , we can approximate them using a Lagrange interpolating polynomial or using a refined mesh only near at their occurrences, see Section 3.2, and see Figure 3, Figure 4.

For a channel problem, see [3], if

$$(5.3a) \quad \int_0^1 f(s, y) ds = 0,$$

and

$$(5.3b) \quad f(x, y) \text{ is 1- periodic in } x,$$

ordinary boundary layers are suppressed. In the Dirichlet boundary value problem, if (5.3a) holds, then the ordinary boundary layers will be mild; note that  $u^0(0, y) = 0$  from (2.4a) and hence from (2.25) we find

$$(5.4) \quad |\theta^0|_{L^2} \leq \kappa \epsilon^{3/4}, \quad |\partial \theta^0 / \partial x|_{L^2} \leq \kappa \epsilon^{-1/4}, \quad |\partial \theta^0 / \partial y|_{L^2} \leq \kappa \epsilon^{1/4}.$$

We thus summarize the occurrences of the boundary layers and their norm estimates.

• Occurrences of boundary layers

Dirichlet boundary problem			mixed boundary problem	channel problem
$u = 0$ on $\partial\Omega$			$\begin{cases} u = 0 \text{ at } x = 0, 1 \\ \partial u / \partial y = 0 \text{ at } y = 0, 1 \end{cases}$	$\begin{cases} u = 0 \text{ at } y = 0, 1 \\ u \text{ is 1- periodic in } x \end{cases}$
conditions on $f$	$f = 0$ at $y = 0, 1$ .	no conditions	compatibility (4.20)	$\int_0^1 f(s, y) ds = 0$ $f$ is 1- periodic in $x$
OBLs	YES at $x = 0$	YES at $x = 0$	YES at $x = 0$	NO
PBLs	NO	YES at $y = 0, 1$	Mild PBLs at $y = 0, 1$ $\sim \begin{cases} O(\epsilon^{3/4}) \text{ in } L^2 \\ O(\epsilon^{1/4}) \text{ in } H^1 \end{cases}$	YES at $y = 0, 1$

• Norm estimates

	O B L	P B L	OBL element	PBL elements
	$\theta^j$	$\varphi^j \sim \varphi_l^j + \varphi_u^j$	$\phi_0$	$\psi_l^j, \psi_u^j, \psi_0^j, \psi_N^j$
local (or, stretched) variables	$\bar{x} = x/\epsilon$	$\bar{y} = y/\sqrt{\epsilon}, \tilde{y} = (1-y)/\sqrt{\epsilon}$	.	.
$L^2$ - norm	$O(\epsilon^{1/2})$	$O(\epsilon^{1/4})$	$O(1)$	$O(1)$
$H^1$ - norm	$O(\epsilon^{-1/2})$	$O(\epsilon^{-1/4})$	$O(\epsilon^{-1/2})$	$O(\epsilon^{-1/4})$

## 6. Numerical Simulations

In this section we present the numerical results of the new scheme (NS) and, the new scheme with PBLs (NSP), which are corresponding to the problems (3.25) and (3.27). Each uses  $V_N$ , and  $\tilde{V}_N$ , respectively, as the finite element spaces.

The numerical calculations were carried out on 4CPU POWER3+ 375 Mhz with 2GB memory running AIX. We solve the linear system directly by using Gaussian elimination. The numerical simulations for the mixed boundary value problem (1.1a)(1.1c) with its approximating system (4.34) have been shown in [1].

**6.1. Numerical Implementations.** To compute the approximating systems (3.25), (3.27) and (4.34), we must evaluate integrations of singular functions, which are ordinary and parabolic boundary layer elements. Since our approximating systems have a small coercivity ( $= \epsilon$ ), we need to take care of computations involving  $\epsilon$  with a caution; the explicit integration formulas are available using MAPLE, and in particular, the integrations involving  $\phi_0$ . We would like to mention that even after we get those explicit formulas, we have to modify them as follows to avoid the overflow of numeric limits in computers, e.g., if  $\epsilon = 10^{-4}$ ,  $e^{1/\epsilon} = e^{10000}$  is out of the range of a double precision. For example, in the formula we have the following terms which need to be modified as below:

$$(6.1) \quad \frac{1}{e^{h_1/\epsilon}} \rightarrow e^{-h_1/\epsilon}, \quad \frac{e^{-h_1/\epsilon}}{\epsilon^8} \rightarrow \left( \frac{e^{-h_1/(8\epsilon)}}{\epsilon} \right)^8.$$

**6.2. Numerical Results : Examples.** For the function  $f = \sin(\pi y)$ , we have tested several cases of  $M$ ,  $N$  and  $\epsilon$ , and obtained  $L^2$ -, and  $L^\infty$ - errors in [1]. This function  $f$  clearly satisfies the condition  $f(x, 0) = f(x, 1) = 0$ , and hence the parabolic boundary layer will not appear. Then we see that the ordinary boundary layer element  $\phi_0$  will play an important role to approximate solutions as indicated in previous sections. For the classical schemes, i.e., without boundary layer elements, we have

observed wild oscillations in the  $x$ - direction, i.e., in the propagation direction due to the convective term  $u_x$ . But the ordinary boundary layer element  $\phi_0$  stabilizes the approximate solutions and thus the approximation errors, see [1] and Figure 2 below for the classical and new schemes. In particular, we have simulated for  $f = y(1 - y)$  and  $f = x^2(1 - e^{-y})(1 - e^{-(1-y)})$  in Figure 2 which satisfy  $f(x, 0) = f(x, 1) = 0$ .

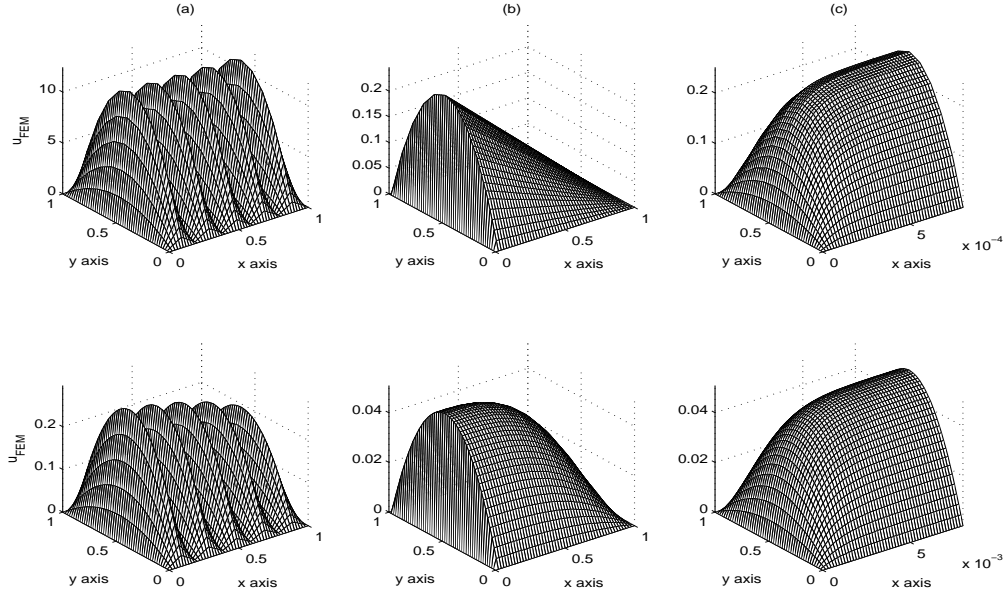


FIGURE 2.  $-\epsilon \Delta u - u_x = f$ ,  $f = y(1 - y)$  in the first row with Dirichlet boundary condition,  $\epsilon = 10^{-4}$ ,  $M = 10$ ,  $N = 10$ ,  $f = x^2(1 - e^{-y})(1 - e^{-(1-y)})$  in the second row with Dirichlet boundary condition,  $\epsilon = 10^{-3}$ ,  $M = 10$ ,  $N = 20$  : (a) Classical Scheme without BL elements; (b) New Scheme (3.25) with OBL element  $\phi_0$ ; (c) Zooming of (b) near  $x = 0$ .

In Figure 3, we notice, for  $f = \cos(\pi y)$  and  $\epsilon = 10^{-6}$ , that the parabolic boundary layers appear in (a) at  $y = 0$  and  $y = 1$ ; note that  $f(x, 0) \neq 0$  and  $f(x, 1) \neq 0$ . But since the propagation direction, i.e., the  $x$ - direction, is parallel to the place of the occurrences of parabolic boundary layers, the approximation errors due to them are localized near  $y = 0$  and  $y = 1$  as shown in (a) of Figure 3 and in (a) of Figure 4,



see also [1] for the errors due to PBLs in numerical simulations. To resolve these errors, as in (b) we added the parabolic boundary layer elements. More precisely, we decompose  $f = \cos(\pi y)$  as below: since  $f(x, 0) = 1$  and  $f(x, 1) = -1$ ,

$$(6.2a) \quad f = f_1 + f_2,$$

where

$$(6.2b) \quad f_1 = \cos(\pi y) - f_2,$$

$$(6.2c) \quad \begin{aligned} f_2 &= f(x, 0)(1 - y) + f(x, 1)y \\ &= 1 - 2y. \end{aligned}$$

Hence, we find from  $f_2$ ,

$$(6.3) \quad u^* = \psi_l^0 - \psi_u^0 + \psi_0^0 \phi_0 - \psi_N^0 \phi_0,$$

and the approximate solution  $u_N^*$  as in (3.27):

$$(6.4) \quad u_N^* = u_N + u^*,$$

where  $u_N$  is the solution of the approximating system (3.25) with  $f_1$  in (6.2b). Then as indicated in Theorem 4.2, we can expect that the approximation errors due to PBLs is  $O(\epsilon^{3/4}) = O(10^{-9/2})$  in  $L^2$ . In Figure 3, one might notice that the approximate solution  $u_N^*$  is composed of (d), (e), and (f), and we see that the PBL elements and the overlappings of PBL and OBL elements in (e) contribute to the approximate solution the most, the OBL elements in (d) the next, and the classical elements in (f) the least.

If  $f(x, 0)$  and  $f(x, 1)$  are not polynomials, we have to allow a truncation error  $f_3$  which is defined in (3.3). In Figure 4, we have tested the case for  $f = \exp(x + y)$ . For  $f_2$ , we use a Lagrange interpolating polynomial of degree 3 defined below. We write

$$(6.5a) \quad f = f_1 + f_2 + f_3,$$

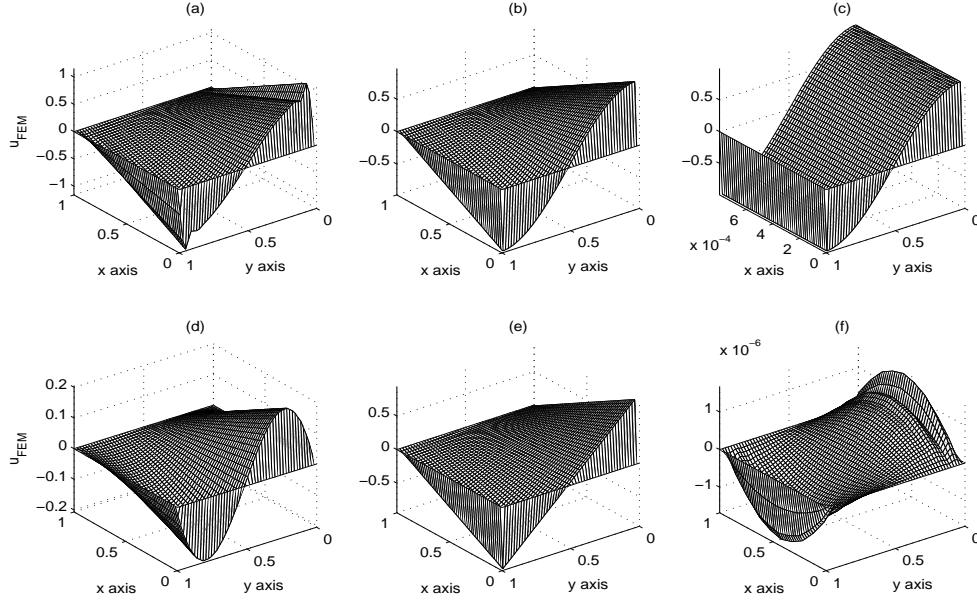


FIGURE 3.  $-\epsilon\Delta u - u_x = \cos(\pi y)$ , with Dirichlet boundary condition,  $\epsilon = 10^{-6}$ ,  $M = 10$ ,  $N = 20$ : (a) New Scheme (3.25) with only OBL element,  $f_1$  being replaced by  $\cos(\pi y)$ ; (b) New Scheme (3.27) with OBL and PBL elements; (c) Zooming of (b) near  $x = 0$ ; (b) = (d) + (e) + (f) where (d) OBL elements  $\phi_0\psi_j$ , (e) PBL elements  $\psi_l^0$ ,  $\psi_u^0$ , and the overlappings of PBL and OBL elements  $\psi_0^0\phi_0$ ,  $\psi_N^0\phi_0$ , and (f) classical elements (or bilinear elements)  $\phi_i\psi_j$ .

where

$$(6.5b) \quad f_1 = \exp(x + y) - f_2 - f_3,$$

$$(6.5c) \quad f_2 = f(x, 0)(1 - y) + f(x, 1)y,$$

with

$$(6.5d) \quad f(x, 0) = \sum_{k=0}^3 \exp(x_k) L_k(x), \quad f(x, 1) = \sum_{k=0}^3 \exp(x_k + 1) L_k(x),$$

where, for  $k = 0, \dots, 3$ ,

$$(6.5e) \quad x_k = \frac{1}{2} \left[ \cos \left( \frac{2k+1}{8} \pi \right) + 1 \right],$$

$$(6.5f) \quad L_k(x) = \prod_{i=0, i \neq k}^3 \frac{(x - x_i)}{(x_k - x_i)},$$

and  $f_3$  is a truncation error: see Lemma 3.3 and Corollary 3.1 for Chebyshev node points  $x_k$ . Then we easily find using e.g. MAPLE,

$$f_2 = (0.27823967x^3 + 0.42430102x^2 + 1.01563252x + 0.9995086147)(1 + 1.718281828y),$$

and hence we can easily derive  $u^*$  similarly as before. It is not hard to find the truncation error,  $\max_{x \in [0,1], y \in \{0,1\}} |f_3| = O(10^{-3})$ , and the approximation errors due to  $f_3$  is also  $O(10^{-3})$  in  $L^2$ . From Theorem 4.2, we can expect that the approximation errors due to PBLs is  $O(\epsilon^{3/4}) = O(10^{-15/4})$  in  $L^2$ . We also notice in Figure 4 that  $u_N^*$  is composed of (d), (e), (f), and (g), and we see that the overlappings of PBL and OBL elements in (f) contribute to the approximate solution the most, the OBL and PBL elements in (d)(e) the next, and the classical elements in (g) the least.

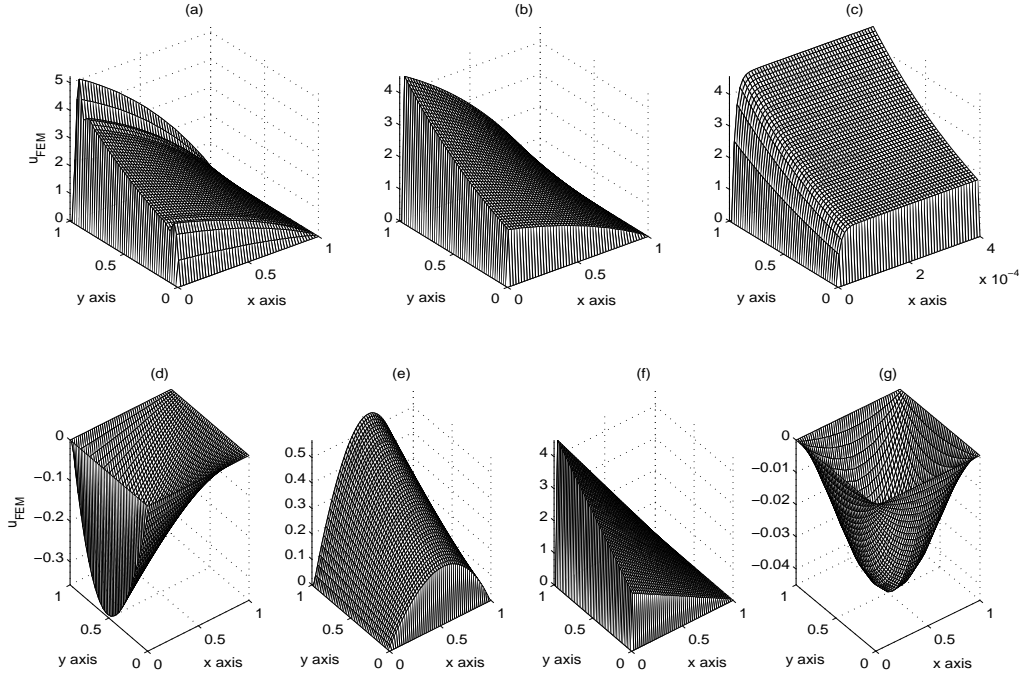


FIGURE 4.  $-\epsilon \Delta u - u_x = \exp(x + y)$ , with Dirichlet boundary condition,  $\epsilon = 10^{-5}$ ,  $M = 10$ ,  $N = 20$ : (a) New Scheme (3.25) with only OBL element,  $f_1$  being replaced by  $\exp(x + y)$ ; (b) New Scheme (3.27) with OBL and PBL elements; (c) Zooming of (b) near  $x = 0$ ; (b) = (d) + (e) + (f) + (g) where (d) OBL elements  $\phi_0 \psi_j$ , (e) PBL elements  $\psi_l^n, \psi_u^n$ ,  $n = 0, \dots, 3$ , (f) the overlappings of PBL and OBL elements  $\psi_0^n \phi_0, \psi_N^n \phi_0$ ,  $n = 0, \dots, 3$ , and (g) classical elements (or bilinear elements)  $\phi_i \psi_j$ .

## 7. Appendix

PROOF OF LEMMA 2.1. By a simple change of variable, we can derive the solution  $u$  of equation (2.8) as follows. Let

$$(7.1) \quad \tilde{x} = 1 - x.$$

We then have the following classical heat equation in the semi-strip  $D = (0, 1) \times (0, \infty)$ :

$$(7.2a) \quad \frac{\partial u}{\partial \tilde{x}} - \frac{\partial^2 u}{\partial y^2} = f^*(1 - \tilde{x}, y), \text{ for } (\tilde{x}, y) \in D,$$

$$(7.2b) \quad u(\tilde{x}, 0) = g^*(1 - \tilde{x}), \quad 0 < \tilde{x} < 1,$$

$$(7.2c) \quad u(\tilde{x}, y) \rightarrow 0 \text{ as } y \rightarrow \infty, \quad 0 < \tilde{x} < 1,$$

$$(7.2d) \quad u(\tilde{x} = 0, y) = 0.$$

From Theorem 20.3.1 in [8], see also [28], we find that the solution  $u$  is unique and is explicitly expressed as:

$$(7.3a) \quad u(\tilde{x}, y) = I_1 + I_2,$$

with

$$(7.3b) \quad I_1 = -2 \int_0^{\tilde{x}} \frac{\partial K}{\partial y}(\tilde{x} - \tau, y) g^*(1 - \tau) d\tau,$$

$$(7.3c) \quad I_2 = \int_0^{\tilde{x}} \int_0^\infty G(\tilde{x} - \tau, y; t) f^*(1 - \tau, t) dt d\tau,$$

where the fundamental solution  $K = K(\tilde{x}, y)$  and the Green function  $G = G(\tilde{x}, y; t)$  are

$$(7.3d) \quad K(\tilde{x}, y) = \frac{1}{\sqrt{4\pi\tilde{x}}} \exp\left(-\frac{y^2}{4\tilde{x}}\right),$$

$$(7.3e) \quad G(\tilde{x}, y; t) = K(\tilde{x}, y - t) - K(\tilde{x}, y + t).$$

Then, since

$$(7.4) \quad \frac{\partial}{\partial y} K(\tilde{x} - \tau, y) = -\frac{y}{4\sqrt{\pi}(\tilde{x} - \tau)^{3/2}} \exp\left(-\frac{y^2}{4(\tilde{x} - \tau)}\right),$$

we find

$$(7.5a) \quad \begin{aligned} I_1 &= \frac{y}{2\sqrt{\pi}} \int_0^{\tilde{x}} \frac{1}{(\tilde{x} - \tau)^{3/2}} \exp\left(-\frac{y^2}{4(\tilde{x} - \tau)}\right) g^*(1 - \tau) d\tau \\ &= (\text{setting } t = y/\sqrt{2(\tilde{x} - \tau)}, \tilde{x} = 1 - x) \\ &= \sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) g^*\left(x + \frac{y^2}{2t^2}\right) dt, \end{aligned}$$

and we also find

$$(7.5b) \quad \begin{aligned} I_2 &= \int_0^{\tilde{x}} \int_0^{\infty} \frac{1}{\sqrt{4\pi(\tilde{x} - \tau)}} \left\{ \exp\left[-\frac{(y-t)^2}{4(\tilde{x} - \tau)}\right] - \exp\left[-\frac{(y+t)^2}{4(\tilde{x} - \tau)}\right] \right\} f^*(1 - \tau, t) dt d\tau \\ &= (\text{setting } s = \tilde{x} - \tau, \tilde{x} = 1 - x) \\ &= \frac{1}{2\sqrt{\pi}} \int_0^{1-x} \int_0^{\infty} \frac{1}{\sqrt{s}} \left\{ \exp\left[-\frac{(y-t)^2}{4s}\right] - \exp\left[-\frac{(y+t)^2}{4s}\right] \right\} f^*(x + s, t) dt ds; \end{aligned}$$

hence (2.13) follows.

We will see that if conditions (2.9) hold, from (7.7), (7.12), (7.15), (7.17), and (7.23), since  $u$  is  $l + 1$  times differentiable on  $\bar{D}$ , we easily find that  $u \in C^l(\bar{D})$ <sup>4</sup>.

We now prove the estimates (2.14) and (2.16); we perform calculations similar to those in the proof of Theorem 3.8 in [28]. For  $m = 0$ , thanks to (2.9) and the explicit form of  $u$  in (2.13), we find

$$(7.7) \quad \frac{\partial^i}{\partial x^i} I_1 = \sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) \frac{\partial^i}{\partial x^i} g^*\left(x + \frac{y^2}{2t^2}\right) dt, \text{ for } 0 \leq i \leq l + 1.$$

---

<sup>4</sup>If  $g^*$  does not satisfy (2.9b), e.g.  $g^*(1) \neq 0$  with  $f^* = 0$ , we then find

$$(7.6) \quad \frac{\partial u}{\partial x} = \sqrt{\frac{2}{\pi}} \left[ \int_{y/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) g^{*'}\left(x + \frac{y^2}{2t^2}\right) dt - y(2(1-x))^{-3/2} g^*(1) \exp\left(-\frac{y^2}{4(1-x)}\right) \right],$$

see [28]. Note that the second term in (7.6) does not belong to  $C^0(\bar{\Omega})$ .

Since  $g^*$  is sufficiently smooth and  $0 \leq x + y^2/(2t^2) \leq 1$ , we have

$$(7.8) \quad \left| \frac{\partial^i}{\partial x^i} g^* \left( x + \frac{y^2}{2t^2} \right) \right| \leq \kappa,$$

and since

$$(7.9) \quad \exp \left( -\frac{t^2}{2} \right) \leq \exp \left( \frac{c^2}{2} - ct \right) = \exp \left( \frac{c^2}{2} \right) \exp(-ct), \text{ for any } c > 0,$$

we can write

$$(7.10) \quad \exp \left( -\frac{t^2}{2} \right) \leq \kappa(c) \exp(-ct), \text{ for any } c > 0.$$

Hence

$$(7.11) \quad \begin{aligned} \left| \frac{\partial^i}{\partial x^i} I_1 \right| &\leq \kappa(c) \int_{y/\sqrt{2(1-x)}}^{\infty} \exp(-ct) dt \leq \kappa(c) \exp \left( -\frac{cy}{\sqrt{2}} \right) \\ &\leq (\text{setting } c = \sqrt{2}\gamma) \leq \kappa(\gamma) \exp(-\gamma y). \end{aligned}$$

We can also differentiate  $I_2$  using (2.15) as follows: for  $0 \leq i \leq l+1$ ,

$$(7.12) \quad \frac{\partial^i}{\partial x^i} I_2 = \frac{1}{2\sqrt{\pi}} \int_0^{1-x} \int_0^{\infty} \frac{1}{\sqrt{s}} \left\{ \exp \left[ -\frac{(y-t)^2}{4s} \right] - \exp \left[ -\frac{(y+t)^2}{4s} \right] \right\} \frac{\partial^i}{\partial x^i} f^*(x+s, t) dt ds.$$

Then since  $0 \leq s \leq 1$ , from the condition (2.15) we find

$$(7.13) \quad \begin{aligned} \left| \frac{\partial^i}{\partial x^i} I_2 \right| &\leq \kappa \left[ \int_0^{1-x} \frac{1}{\sqrt{s}} ds \right] \int_0^{\infty} \exp \left( -\frac{(y-t)^2}{4} \right) \exp(-\gamma t) dt \\ &\leq \kappa \int_0^{\infty} \exp \left( -\frac{(t + (2\gamma - y))^2}{4} - \gamma y + \gamma^2 \right) dt \\ &\leq \kappa \exp(-\gamma y) \int_0^{\infty} \exp \left( -\frac{(t + (2\gamma - y))^2}{4} \right) dt \leq \kappa \exp(-\gamma y). \end{aligned}$$

Hence, from (7.11) and (7.13), we conclude that

$$(7.14) \quad \left| \frac{\partial^i}{\partial x^i} u(x, y) \right| \leq \kappa \exp(-\gamma y), \forall (x, y) \in D,$$

for  $0 \leq i \leq l+1$ .

For  $m \geq 1$ , we treat repeatedly the case where  $m = 2k$  and where  $m = 2k + 1$ . Firstly, for  $m = 2k$ ,  $k \geq 1$ , differentiating equation (2.10)  $i$  times in  $x$ , we then find

$$(7.15) \quad \frac{\partial^{i+2k}}{\partial x^i \partial y^{2k}} u = (-1)^k \frac{\partial^{i+k}}{\partial x^{i+k}} u + \sum_{s=0}^{k-1} (-1)^{s+1} \frac{\partial^{i+2k-s-2}}{\partial x^{i+s} \partial y^{2(k-s-1)}} f^*;$$

note that  $i + k \leq l + 1 - k \leq l + 1$ , and hence

$$(7.16) \quad \left| \frac{\partial^{i+2k}}{\partial x^i \partial y^{2k}} u \right| \leq \left| \frac{\partial^{i+k}}{\partial x^{i+k}} u \right| + \sum_{s=0}^{k-1} \left| \frac{\partial^{i+2k-s-2}}{\partial x^{i+s} \partial y^{2(k-s-1)}} f^* \right|.$$

We then easily find from (2.15) and (7.14) that the estimates (2.16) with  $m = 2k$  hold.

If  $m = 2k + 1$ ,  $k \geq 0$ , we proceed as follows. Firstly we find similarly

$$(7.17) \quad \begin{aligned} \frac{\partial^{1+i}}{\partial x^i \partial y} I_1 &= \frac{\partial^{1+i}}{\partial y \partial x^i} I_1 \\ &= \sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) \frac{\partial^{1+i}}{\partial x^{1+i}} g^*\left(x + \frac{y^2}{2t^2}\right) \frac{y}{t^2} dt, \end{aligned}$$

for  $0 \leq i \leq l$ . Since

$$(7.18) \quad \left| \frac{\partial^{1+i}}{\partial x^{1+i}} g^*\left(x + \frac{y^2}{2t^2}\right) \right| \leq \kappa,$$

we find

$$(7.19) \quad \begin{aligned} \left| \frac{\partial^{1+i}}{\partial y \partial x^i} I_1 \right| &\leq \kappa y \int_{y/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) d(-t^{-1}) \\ &\leq (\text{integrating by parts}) \\ &\leq \kappa \exp\left(-\frac{y^2}{4}\right) + \kappa y \int_{y/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt \\ &\leq \kappa \exp\left(-\frac{(y-1)^2 - 1}{4}\right) \exp\left(-\frac{y}{2}\right) \\ &\quad + \kappa y \int_{y/\sqrt{2(1-x)}}^{\infty} \exp\left(-\frac{(t-1)^2 - 1}{2}\right) \exp(-t) dt \\ &\leq \kappa \exp\left(-\frac{y}{2}\right) + \kappa y \int_{y/\sqrt{2(1-x)}}^{\infty} \exp(-t) dt \\ &\leq \kappa \exp\left(-\frac{y}{2}\right) + \kappa y \exp\left(-\frac{y}{\sqrt{2}}\right) \leq \kappa \exp\left(-\frac{y}{2}\right), \end{aligned}$$



for  $0 \leq i \leq l$ . We also find

$$(7.20) \quad \frac{\partial^{i+1}}{\partial x^i \partial y} I_2 = \frac{\partial^{i+1}}{\partial y \partial x^i} I_2 = \frac{1}{2\sqrt{\pi}} \int_0^{1-x} \int_0^\infty \frac{1}{\sqrt{s}} \frac{\partial}{\partial y} \left\{ \exp \left[ -\frac{(y-t)^2}{4s} \right] - \exp \left[ -\frac{(y+t)^2}{4s} \right] \right\} \frac{\partial^i}{\partial x^i} f^*(x+s, t) dt ds,$$

and

$$(7.21) \quad \left| \frac{\partial^{1+i}}{\partial y \partial x^i} I_2 \right| \leq \kappa \left[ \int_0^{1-x} \frac{1}{\sqrt{s}} \cdot I_3 ds \right],$$

where

$$\begin{aligned} I_3 &= \left| \int_0^\infty \frac{\partial}{\partial y} \left\{ \exp \left[ -\frac{(y-t)^2}{4s} \right] - \exp \left[ -\frac{(y+t)^2}{4s} \right] \right\} \frac{\partial^i}{\partial x^i} f^*(x+s, t) dt \right| \\ &= \left| \int_0^\infty \frac{\partial}{\partial t} \left\{ -\exp \left[ -\frac{(y-t)^2}{4s} \right] - \exp \left[ -\frac{(y+t)^2}{4s} \right] \right\} \frac{\partial^i}{\partial x^i} f^*(x+s, t) dt \right| \\ &= (\text{integrating by parts}) \leq 2 \exp \left[ -\frac{y^2}{4s} \right] \left| \frac{\partial^i}{\partial x^i} f^*(x+s, 0) \right| \\ &\quad + \int_0^\infty \left\{ \exp \left[ -\frac{(y-t)^2}{4s} \right] + \exp \left[ -\frac{(y+t)^2}{4s} \right] \right\} \left| \frac{\partial^{1+i}}{\partial t \partial x^i} f^*(x+s, t) \right| dt \\ &\leq (\text{by (2.15)}) \leq \kappa \exp(-\gamma y) + \kappa \int_0^\infty \exp \left( -\frac{(y-t)^2}{4} \right) \exp(-\gamma t) dt \\ &\leq (\text{similarly to (7.13)}) \leq \kappa \exp(-\gamma y). \end{aligned}$$

Hence, we conclude

$$(7.22) \quad \left| \frac{\partial^{1+i}}{\partial y \partial x^i} u \right| \leq \kappa \exp(-\gamma y), \text{ for } 0 \leq i \leq l.$$

Differentiating (7.15) in  $y$ , we easily find

$$(7.23) \quad \frac{\partial^{i+2k+1}}{\partial x^i \partial y^{2k+1}} u = (-1)^k \frac{\partial^{i+k+1}}{\partial y \partial x^{i+k}} u + \sum_{s=0}^{k-1} (-1)^{s+1} \frac{\partial^{i+2k-s-1}}{\partial x^{i+s} \partial y^{2(k-s)-1}} f^*;$$

note that  $i+k \leq l-k \leq l$ , and hence

$$(7.24) \quad \left| \frac{\partial^{i+2k+1}}{\partial x^i \partial y^{2k+1}} u \right| \leq \left| \frac{\partial^{i+k+1}}{\partial y \partial x^{i+k}} u \right| + \sum_{s=0}^{k-1} \left| \frac{\partial^{i+2k-s-1}}{\partial x^{i+s} \partial y^{2(k-s)-1}} f^* \right|.$$

We then easily find from (2.15) and (7.22) that the estimates (2.16) with  $m = 2k+1$  hold. Hence, the estimates (2.16) follow.  $\square$

PROOF OF LEMMA 4.1. As in the proof of Lemma 2.1, we set

$$(7.25) \quad \tilde{x} = 1 - x.$$

Then we have the following classical heat equation in a semi-strip  $D$ :

$$(7.26a) \quad \frac{\partial u}{\partial \tilde{x}} - \frac{\partial^2 u}{\partial y^2} = \epsilon^{1/2} f^*(1 - \tilde{x}, y), \text{ for } (\tilde{x}, y) \in D,$$

$$(7.26b) \quad \frac{\partial u}{\partial y}(\tilde{x}, 0) = \epsilon^{1/2} g^*(1 - \tilde{x}), \quad 0 < \tilde{x} < 1,$$

$$(7.26c) \quad u(\tilde{x}, y) \rightarrow 0 \text{ as } y \rightarrow \infty, \quad 0 < \tilde{x} < 1,$$

$$(7.26d) \quad u(\tilde{x} = 0, y) = 0.$$

From Theorem 20.3.2 in [8], we find that the solution  $u$  is unique and it is explicitly expressed as:

$$(7.27a) \quad u(\tilde{x}, y) = I_4 + I_5,$$

with

$$(7.27b) \quad I_4 = -2\epsilon^{1/2} \int_0^{\tilde{x}} K(\tilde{x} - \tau, y) g^*(1 - \tau) d\tau,$$

$$(7.27c) \quad I_5 = \epsilon^{1/2} \int_0^{\tilde{x}} \int_0^\infty N(\tilde{x} - \tau, y; t) f^*(1 - \tau, t) dt d\tau,$$

where the fundamental solution  $K = K(\tilde{x}, y)$  of the heat equation is as in (7.3d) and the Green function  $N = N(\tilde{x}, y; t)$  is

$$(7.27d) \quad N(\tilde{x}, y; t) = K(\tilde{x}, y - t) + K(\tilde{x}, y + t).$$

Then

$$\begin{aligned} (7.28a) \quad I_4 &= -2\epsilon^{1/2} \int_0^{\tilde{x}} \frac{1}{\sqrt{4\pi(\tilde{x} - \tau)}} \exp\left(-\frac{y^2}{4(\tilde{x} - \tau)}\right) g^*(1 - \tau) d\tau \\ &= (\text{setting } t = \tilde{x} - \tau, \tilde{x} = 1 - x) \\ &= -\frac{\epsilon^{1/2}}{\sqrt{\pi}} \int_0^{1-x} \frac{1}{\sqrt{t}} \exp\left(-\frac{y^2}{4t}\right) g^*(x + t) dt; \end{aligned}$$

we also find

(7.28b)

$$\begin{aligned}
I_5 &= \epsilon^{1/2} \int_0^{\tilde{x}} \int_0^\infty \frac{1}{\sqrt{4\pi(\tilde{x} - \tau)}} \left\{ \exp \left[ -\frac{(y - t)^2}{4(\tilde{x} - \tau)} \right] + \exp \left[ -\frac{(y + t)^2}{4(\tilde{x} - \tau)} \right] \right\} f^*(1 - \tau, t) dt d\tau \\
&= (\text{setting } s = \tilde{x} - \tau, \tilde{x} = 1 - x) \\
&= \frac{\epsilon^{1/2}}{2\sqrt{\pi}} \int_0^{1-x} \int_0^\infty \frac{1}{\sqrt{s}} \left\{ \exp \left[ -\frac{(y - t)^2}{4s} \right] + \exp \left[ -\frac{(y + t)^2}{4s} \right] \right\} f^*(x + s, t) dt ds;
\end{aligned}$$

hence (4.9) follows.

Similarly to the proof in Lemma 2.1, we find that conditions (4.6) are sufficient for  $u \in C^l(\bar{D})$  and for the estimates (4.10) and (4.12), we can also proceed similarly; we use here the fact that since  $g^*$  is sufficiently smooth,

$$\begin{aligned}
(7.29) \quad \left| \int_0^{1-x} \frac{1}{\sqrt{t}} \exp \left( -\frac{y^2}{4t} \right) \frac{\partial^i}{\partial x^i} g^*(x + t) dt \right| &\leq \kappa \left[ \int_0^1 \frac{1}{\sqrt{t}} dt \right] \exp \left( -\frac{y^2}{4} \right) \\
&\leq (\text{by (7.10)}) \leq \kappa \exp(-\gamma y),
\end{aligned}$$

and

$$(7.30) \quad \frac{\partial}{\partial y} I_4 = \epsilon^{1/2} I_1,$$

where  $I_1$  is as in (7.3b). □

## CHAPTER 3

### $L^2$ -Stability Analysis

#### 1. Introduction

In this chapter we consider linear singularly perturbed boundary value problems of the types:

$$(1.1a) \quad -\epsilon \Delta u^\epsilon - u_x^\epsilon = f \text{ in } \Omega,$$

$$(1.1b) \quad u^\epsilon = 0 \text{ on } \partial\Omega,$$

where  $0 < \epsilon \ll 1$ ,  $\Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$ . The function  $f$  is assumed to be smooth on  $\bar{\Omega}$  but we will assume only in Section 3 below (for the  $L^2$ - stability analysis) that  $f$  belongs to only  $L^2(\Omega)$ .

As  $\epsilon$  becomes small, the solutions to such problems generally display near the boundaries, thin transition layers called boundary layers, which are due to the fact that the boundary conditions of the problem are not the same for  $\epsilon > 0$  and  $\epsilon = 0$ , and then (for  $\epsilon > 0$  small) certain derivatives of the solutions become very large near the boundaries. We expect that near those boundary layers, the approximation errors of the discretized system corresponding to problem (1.1) become very large (due to the large  $H^2$ - singularities of the boundary layers). When the stiffness of this discretized system is not properly handled, those large approximation errors at the boundaries propagate in the whole domain due to the convective term, e.g.  $-u_x$  in (1.1a), and then the numerical solutions show a highly oscillatory behavior, see Chapter 2, e.g. [6], [9], [10], [17], [1], [2], [25] and [26]. Resolving boundary layers by the classical approximation methods requires very fine meshes, which is costly to

realize in practice. Indeed, the thickness of the boundary layers (of order  $O(\epsilon)$  for ordinary boundary layers (OBL), and of order  $O(\epsilon^{1/2})$  for parabolic boundary layers (PBL), see Chapter 2) is much smaller than the mesh size  $\bar{h}$ . Notice that our problem (1.1) produces both OBLs at  $x = 0$  and PBLs at  $y = 0, 1$  which pollute, respectively, globally and locally the numerical solutions. Here we impose the condition

$$(1.2) \quad f(x, 0) = f(x, 1) = 0$$

so that the PBLs are suppressed. Indeed if (1.2) is not satisfied, we observe PBLs, but we showed in Chapter 2 how one can handle the PBLs independently of the discretizations by using a Lagrange interpolating polynomial for  $f(x, 0)$  and  $f(x, 1)$ . Hence (1.2) is assumed here.

In Chapter 2 or [1], [2] and [9], it is demonstrated that the boundary layer elements (BLE) play an essential role in the finite element approximations for singularly perturbed problems producing the OBLs. However, the BLE, corresponding to (1.1) and constructed in Chapter 2 or [9], namely  $\phi_0^* = -\exp(-x/\epsilon) - (1 - \exp(-1/\epsilon))x + 1$ , does not have a compact support and this leads to a broad band in the stiffness matrix and hence the corresponding systems are costly to solve. Thus, a first aim in this chapter is to revise the element  $\phi_0^*$  so that it has a small compact support and to prove that the numerical approximations keep the same accuracy as before. Then the new system appears to be sparse, more precisely, block tridiagonal and it can be solved very efficiently; it requires essentially the same computing resources as those in the classical methods which use only classical elements, e.g.  $Q_1, Q_2$ . Furthermore, since the stiffness matrix is tridiagonal, via a somehow involved matrix analysis we are able to analyze the stability in the  $L^2$ - norm; we will prove that for any  $f \in L^2$ ,  $|u_N|_{L^2} \leq \kappa |f|_{L^2}$ , where  $u_N$  is an approximate solution, and the positive constant  $\kappa$  is independent of the mesh size  $\bar{h}$  and of the small parameter  $\epsilon$ , see Section 3 below.

In summary we show that our schemes are very efficient and economical though they only require a space mesh of order  $\bar{h} = o(\epsilon^{1/2})$  for a suitable  $H^1$  approximation and of order  $\bar{h} = o(\epsilon^{1/4})$  for a suitable  $L^2$  approximation..

Here we denote the mesh size by  $\bar{h} = \max\{h_1, h_2\}$  where  $h_1 = 1/M$ ,  $h_2 = 1/N$ , and  $M, N$  are the number of elements in the  $x$ -, and  $y$ - directions, respectively. Hence, the number of rectangular elements is  $MN$ .

We shall consider the Sobolev spaces  $H^m(\Omega)$ ,  $m$  integer, equipped with the semi-norm,  $|u|_{H^m} = \left( \sum_{|\alpha|=m} \int_{\Omega} |D^{\alpha}u|^2 d\Omega \right)^{1/2}$ , and the norm,  $\|u\|_{H^m} = \left( \sum_{j=0}^m |u|_{H^j}^2 \right)^{1/2}$ . We also define the corresponding inner product in the space  $H^m(\Omega)$ :  $((u, v))_{H^m} = \sum_{|\alpha| \leq m} (D^{\alpha}u, D^{\alpha}v)$ , where  $(u, v) = \int_{\Omega} uv d\Omega$ . For the Dirichlet boundary value problem (1.1), we use the Sobolev space  $H_0^1(\Omega)$ , which is the closure in the space  $H^1(\Omega)$  of  $C^{\infty}$  functions compactly supported in  $\Omega$ . As usual, when  $m = 0$ , we denote the space by  $L^2$ . In the text  $\kappa$  denotes a generic constant independent of  $\epsilon, h_1, h_2, \bar{h}$ , which may be different at different occurrences. But if it needs to be distinguished, we denote it by  $\kappa_i$ ,  $i = 0, 1, \dots$ , and so on.

To solve the problem (1.1) in the finite element context, we consider its weak formulation: *To find  $u \in H_0^1(\Omega)$  such that*

$$(1.3a) \quad a_{\epsilon}(u, v) = F(v), \quad \forall v \in H_0^1(\Omega),$$

where

$$(1.3b) \quad a_{\epsilon}(u, v) = \epsilon((u, v)) - (u_x, v), \quad F(v) = (f, v);$$

thanks to the Poincaré inequality the space  $H_0^1(\Omega)$  is equipped with the inner product  $((\cdot, \cdot))$ , and the norm  $\|\cdot\|$ , respectively:

$$(1.4) \quad ((u, v)) = \int_{\Omega} \nabla u \cdot \nabla v d\Omega, \quad \|u\| = |u|_{H^1} = ((u, u))^{1/2}.$$

It is then easy to verify the coercivity, because

$$(1.5) \quad a_\epsilon(u, u) \geq \epsilon \|u\|^2, \text{ for all } u \in H_0^1(\Omega).$$

We also easily verify the continuity of the bilinear form  $a_\epsilon$  on  $H_0^1(\Omega) \times H_0^1(\Omega)$  and the continuity of the linear form  $F$  on  $H_0^1(\Omega)$ ; hence, by the Lax-Milgram theorem, there exists a unique function  $u \in H_0^1(\Omega)$  satisfying equation (1.3). Note that the coercivity constant ( $= \epsilon$ ) is very small and hence we will see (in Section 3) that the classical approximation system is highly unstable and ill-conditioned.

This chapter is organized as follows: We start in Section 2 by modifying  $\phi_0^*$  slightly and we construct a new boundary layer element  $\phi_0$  which has a small compact support; this will be used in the stability analysis, and it will be used elsewhere in the numerical simulations. In section 2.2, we consider new finite element schemes using the element  $\phi_0$ ; more precisely, the function  $\phi_0$  will be incorporated into the appropriate finite element space that we will define. We then perform the  $L^2$ - stability analysis via a matrix method in Section 3 and derive error estimates in  $H^1$  and  $L^2$  in Section 4.

A number of technical hypotheses will be needed in  $h_1$ ,  $h_2$  and  $\epsilon$ , namely (H0) to (H5) (see (2.6), (2.7), (3.20), (3.30), (3.45), (3.46)).

## 2. Boundary Layer Elements (BLE)

Starting with

$$(2.1) \quad \phi_0^*(x) = -\exp(-x/\epsilon) - (1 - \exp(-1/\epsilon))x + 1,$$

which belongs to  $C^\infty([0, 1]) \cap H_0^1(0, 1)$ , we first recall the following lemma from Chapter 2 which states that  $\phi_0^*$  absorbs the  $H^2$ - singularity of the boundary layers.

**LEMMA 2.1.** *Assume that the condition (1.2) holds. Then there exist a positive constant  $\kappa$  independent of  $\epsilon$ , and a smooth function  $g = g^\epsilon(y) \in H_0^1(0, 1)$  with*

$|g|_{H^2(0,1)} \leq \kappa$  such that

$$(2.2) \quad \|u^\epsilon - g\phi_0^*\|_{H^2(\Omega)} \leq \kappa.$$

**2.1. Constructing BLEs.** We now slightly modify  $\phi_0^*$ , and derive a new boundary layer element  $\phi_0$  which has a small compact support, see (a) of Figure 1:

$$(2.3a) \quad \phi_0 = \tilde{\phi}_0(x)\chi_{[0,h_1]}(x),$$

where

$$(2.3b) \quad \tilde{\phi}_0(x) = -\exp(-x/\epsilon) - (1 - \exp(-h_1/\epsilon))x/h_1 + 1,$$

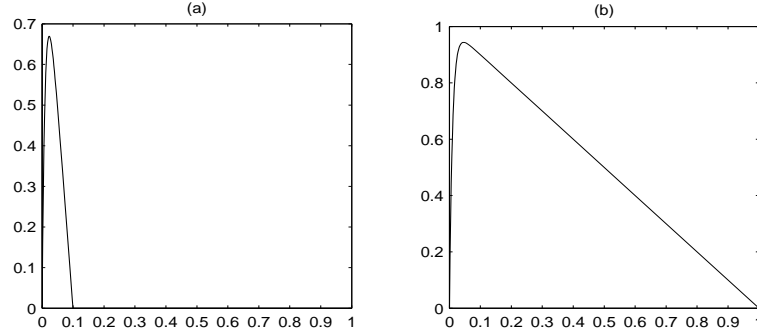


FIGURE 1. Boundary layer elements for  $\epsilon = 0.01$ ,  $h_1 = 0.1$  here; (a):  $\phi_0$ , (b):  $\phi_0^*$ .

To compare the two elements,  $\phi_0^*$  and  $\phi_0$ ,  $\phi_0$  given in (2.3), we rewrite

$$(2.4) \quad \phi_0 = \left[ -\exp\left(-\frac{x}{\epsilon}\right) + \exp\left(-\frac{h_1}{\epsilon}\right) \frac{x}{h_1} \right] \chi_{[0,h_1]} + \left[ 1 - \frac{x}{h_1} \right] \chi_{[0,h_1]}.$$

Since the last term,  $(1 - x/h_1)\chi_{[0,h_1]}$ , is exactly a hat function at  $x = 0$ , we thus easily verify that there exist  $c_i$ 's such that

$$(2.5) \quad \left[ 1 - \frac{x}{h_1} \right] \chi_{[0,h_1]} + \sum_{i=1}^{M-1} c_i \phi_i = 1 - x, \quad \forall x \in [0, 1],$$



where the  $\phi_i$  are the usual hat functions whose definition is recalled in Section 2.2.

We now make two smallness hypotheses for  $\epsilon$ , namely

$$(2.6) \quad (\text{H0}) \quad -\epsilon \ln \epsilon \leq \frac{2}{3}h_1, \left( \text{or } \exp\left(-\frac{h_1}{\epsilon}\right) \leq \epsilon^{3/2} \right),$$

$$(2.7) \quad (\text{H1}) \quad \epsilon \leq \kappa_0 h_1, \quad 0 < \kappa_0 \leq 1/4;$$

these hypotheses simplify calculations here and later and (2.6) will be used to majorize the expressions (3.5h) - (3.5k) below.

LEMMA 2.2. *Assume that (H0) - (2.6), (H1) - (2.7) hold, and let*

$$(2.8) \quad \Phi_0 = \phi_0 + \sum_{i=1}^{M-1} c_i \phi_i,$$

where the  $c_i$ 's are as in (2.5).

Then the following inequalities hold: for  $m = 0, 1$ ,

$$(2.9a) \quad |\phi_0^* - \Phi_0|_{H^m(0,1)} \leq \kappa h_1^m,$$

$$(2.9b) \quad |\Phi_0|_{H^m(0,1)} \leq \kappa \epsilon^{-m/2}.$$

PROOF. Firstly, notice that from (2.1), (2.5) and (2.8)

$$\phi_0^* - \Phi_0 = -\exp\left(-\frac{x}{\epsilon}\right) \chi_{[h_1,1]} - \exp\left(-\frac{h_1}{\epsilon}\right) \frac{x}{h_1} \chi_{[0,h_1]} + \exp\left(-\frac{1}{\epsilon}\right) x;$$

using (2.6) after elementary calculations, (2.9a) follows; note that the term  $\exp(-1/\epsilon)x$  is exponentially small. Then (2.9b) follows from (2.9a) observing that  $|\phi_0^*|_{L^2(0,1)} \leq \kappa$ ,  $|\phi_0^*|_{H^1(0,1)} \leq \kappa \epsilon^{-1/2}$ , and  $|\Phi_0|_{H^m} \leq |\phi_0^*|_{H^m} + |\phi_0^* - \Phi_0|_{H^m}$ .  $\square$

**2.2. Finite Element Discretizations.** We now define the finite element spaces and introduce the new schemes making use of the classical  $Q_1$  elements (hat functions),  $\phi_i$  and  $\psi_j$ ,  $i = 1, \dots, M-1$ ,  $j = 1, \dots, N-1$ , to which we add the boundary layer element (BLE)  $\phi_0$ . We thus introduce the following finite element space:

$$(2.10a) \quad V_N := \left\{ \sum_{j=1}^{N-1} c_{0j} \phi_0 \psi_j + \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} c_{ij} \phi_i \psi_j \right\} \subset H_0^1(\Omega),$$

and we then look for an approximate solution  $u_N \in V_N$  such that

$$(2.10b) \quad a_\epsilon(u_N, v) = F(v), \quad \forall v \in V_N.$$

Note the difference with the classical  $Q_1$  finite element approximation of (1.3) for which the first sum in (2.10a) would not be present.

To derive the  $H^1$ - and  $L^2$ - error estimates below for the scheme (2.10), we will need the following interpolation inequalities.

**LEMMA 2.3.** *Assume that (H0) - (2.6), (H1) - (2.7) hold. Then there exists an interpolant  $\Pi u^\epsilon \in V_N$  such that*

$$(2.11a) \quad \|u^\epsilon - \Pi u^\epsilon\|_{L^2(\Omega)} \leq \kappa \bar{h}^2,$$

$$(2.11b) \quad \|u^\epsilon - \Pi u^\epsilon\|_{H^1(\Omega)} \leq \kappa(\bar{h} + h_2^2 \epsilon^{-1/2}).$$

**PROOF.** From the classical interpolation results, see e.g. [1], [11] and [26] applied to  $u^\epsilon - g\phi_0^* \in H_0^1(\Omega)$ , and by (2.2), we choose the  $c_{ij}$  and  $\kappa$  such that for  $m = 0, 1$ ,

$$I_1(m) := \left\| u^\epsilon - g\phi_0^* - \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} c_{ij} \phi_i \psi_j \right\|_{H^m(\Omega)} \leq \kappa \bar{h}^{2-m} |u^\epsilon - g\phi_0^*|_{H^2(\Omega)} \leq \kappa \bar{h}^{2-m}.$$

By (H0), (H1), we easily find that from (2.9a) for  $m = 0, 1$ ,

$$I_2(m) := \|g\phi_0^* - g\Phi_0\|_{H^m(\Omega)} \leq \kappa h_1^m.$$

Then again, by the classical interpolation results applied this time to  $g = g^\epsilon(y)$ , we can also choose the  $c_{0j}$  and  $\kappa$  such that for  $m = 0, 1$ ,

$$\left\| g - \sum_{j=1}^{N-1} c_{0j} \psi_j \right\|_{H^m(0,1)} \leq \kappa h_2^{2-m};$$

then using the estimates (2.9b), we easily verify the following estimates, observing that  $\Phi_0$  depends only on  $x$ , and  $g$  and  $\psi_j$  depend only on  $y$ :

$$\begin{aligned} I_3(0) &:= \left| g\Phi_0 - \sum_{j=1}^{N-1} c_{0j}\Phi_0\psi_j \right|_{L^2(\Omega)} \leq \left| g - \sum_{j=1}^{N-1} c_{0j}\psi_j \right|_{L^2(0,1)} |\Phi_0|_{L^2(0,1)} \leq \kappa h_2^2, \\ I_3(1) &:= \left| g\Phi_0 - \sum_{j=1}^{N-1} c_{0j}\Phi_0\psi_j \right|_{H^1(\Omega)} \leq \kappa \left| g - \sum_{j=1}^{N-1} c_{0j}\psi_j \right|_{H^1(0,1)} |\Phi_0|_{L^2(0,1)} \\ &\quad + \kappa \left| g - \sum_{j=1}^{N-1} c_{0j}\psi_j \right|_{L^2(0,1)} |\Phi_0|_{H^1(0,1)} \leq \kappa h_2 + \kappa h_2^2 \epsilon^{-1/2}. \end{aligned}$$

The lemma follows after setting

$$\Pi u^\epsilon = \sum_{j=1}^{N-1} c_{0j}\Phi_0\psi_j + \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} c_{ij}\phi_i\psi_j \in V_N,$$

and observing that for  $m = 0, 1$ ,

$$\left\| u^\epsilon - \sum_{j=1}^{N-1} c_{0j}\Phi_0\psi_j - \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} c_{ij}\phi_i\psi_j \right\|_{H^m} \leq I_1(m) + I_2(m) + I_3(m).$$

□

### 3. $L^2$ - stability Analysis

The linear system (2.10) has a unique solution because of the coercivity as observed in (1.5). When  $\epsilon$  is small or  $\epsilon \rightarrow 0$ , one would expect that the linear system is highly ill-conditioned. However, we will show how the new boundary layer element  $\phi_0$  stabilizes (or, absorbs the singularities of) the discretized linear system (2.10). Since the limit system (i.e. when  $\epsilon = 0$ ) has a simple structured block matrix which appears in (3.5) and (3.6) below, we are able to analyze the  $L^2$ - stability via a matrix method.

Setting

$$(3.1) \quad u_N = \sum_{i=0}^{M-1} \sum_{j=1}^{N-1} a_{ij}\phi_i\psi_j,$$

we then write equation (2.10) with  $F(v) = \int_{\Omega} f v d\Omega$ , for any  $f \in L^2$  not necessarily satisfying (1.2), in the matrix from:

$$(3.2) \quad \Gamma_{\epsilon} \mathbf{a} = \mathbf{b}.$$

The  $\Gamma_{\epsilon}$  and the  $\mathbf{b}$  are the stiffness matrix and the load vector specified in (3.5) and (3.42) respectively below, and

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{a}_i \\ \cdot \\ \mathbf{a}_{M-2} \\ \mathbf{a}_{M-1} \end{bmatrix}_{M \times 1}, \quad \mathbf{a}_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \cdot \\ a_{ij} \\ \cdot \\ a_{i,N-1} \end{bmatrix}_{(N-1) \times 1}; \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{b}_k \\ \cdot \\ \mathbf{b}_{M-2} \\ \mathbf{b}_{M-1} \end{bmatrix}_{M \times 1}, \quad \mathbf{b}_k = \begin{bmatrix} b_{k1} \\ b_{k2} \\ \cdot \\ b_{kl} \\ \cdot \\ b_{k,N-1} \end{bmatrix}_{(N-1) \times 1};$$

note that the matrix  $\Gamma_{\epsilon}$  is of size  $[M \times (N-1)]^2 = [(1-h_2)/(h_1 h_2)]^2$ . For the purpose of the analysis below, we introduce the Euclidian inner product  $\langle \cdot, \cdot \rangle$  on  $\mathbb{R}^N$ ,  $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^N a_i b_i$ , where  $\mathbf{a} = (a_1, \dots, a_i, \dots, a_N)^T$ ,  $\mathbf{b} = (b_1, \dots, b_i, \dots, b_N)^T$ . We also introduce the corresponding matrix norm  $\|\Lambda\| = \max_{\|\mathbf{x}\|_2=1} \|\Lambda \mathbf{x}\|_2$ , where  $\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ . We recall the following well-known facts, see [15], [24].

For a matrix  $\Lambda \in \mathbb{R}^{N \times N}$ , setting  $\bar{\rho}(\Lambda) = \max_{\lambda \in \sigma(\Lambda)} |\lambda|$ ,  $\underline{\rho}(\Lambda) = \min_{\lambda \in \sigma(\Lambda)} |\lambda|$  with  $\sigma(\Lambda) = \{\lambda \in \mathbb{C}; \lambda \text{ an eigenvalue of } \Lambda\}$ , we have  $\|\Lambda\| = \{\bar{\rho}(\Lambda^T \Lambda)\}^{1/2}$ , and if  $\Lambda$  is invertible,  $\|\Lambda^{-1}\| = \{\underline{\rho}(\Lambda^T \Lambda)\}^{-1/2}$ , where  $\Lambda^T$  is the transpose of  $\Lambda$ . In particular, if  $\Lambda$  is a symmetric matrix, i.e.  $\Lambda^T = \Lambda$ , then  $\|\Lambda\| = \bar{\rho}(\Lambda)$ ,  $\|\Lambda^{-1}\| = \{\underline{\rho}(\Lambda)\}^{-1}$ .

We will explicitly calculate the entries of the stiffness matrix  $\Gamma_{\epsilon}$ . For that purpose and for the analysis later on, it is convenient here to define the identity matrices,  $I$

and  $\tilde{I}$ , and the tridiagonal matrices, S, U which will be used repeatedly.

$$\begin{aligned}
 I &= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{(N-1) \times (N-1)}, & \tilde{I} &= \begin{bmatrix} I & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix}_{M \times M}, \\
 S &= \begin{bmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 4 \end{bmatrix}_{(N-1) \times (N-1)}, & U &= \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{(N-1) \times (N-1)}.
 \end{aligned}$$

From the Gershgorin circle theorem, we easily find that

$$(3.3) \quad \|S\| \leq 6, \quad \|S^{-1}\| \leq 1/2, \quad \|U\| \leq 4;$$

however for  $\|U^{-1}\|$ , since the Gershgorin discs of U contain the point 0, we are not able to find the upper bound of  $\|U^{-1}\|$ . The relations (3.3) for S follow also from the explicit expression of its eigenvalues which are, see e.g. [7]:  $\lambda_i = 6 - 4 \left( \sin \frac{i\pi}{2N} \right)^2$ ,  $i = 1, \dots, N - 1$ .

We now compute the  $(l, j)^{th}$  entry of each  $(k, i)^{th}$  block,  $a_\epsilon(\phi_i \psi_j, \phi_k \psi_l)$ , of the stiffness matrix  $\Gamma_\epsilon$ :

$$\begin{aligned}
 (3.4) \quad a_\epsilon(\phi_i \psi_j, \phi_k \psi_l) &= \epsilon \int_0^1 \frac{d\phi_i}{dx} \frac{d\phi_k}{dx} dx \int_0^1 \psi_j \psi_l dy \\
 &+ \epsilon \int_0^1 \phi_i \phi_k dx \int_0^1 \frac{d\psi_j}{dy} \frac{d\psi_l}{dy} dy - \int_0^1 \frac{d\phi_i}{dx} \phi_k dx \int_0^1 \psi_j \psi_l dy;
 \end{aligned}$$

the indices  $k, i$  range from 0 to  $M - 1$ , and the indices  $l, j$  range from 1 to  $N - 1$ .

By explicit computation of the integrals in (3.4), we find:

$$(3.5a) \quad \Gamma_\epsilon = \begin{bmatrix} A_\epsilon & B_\epsilon & & & & \\ C_\epsilon & D_\epsilon & E_\epsilon & & & \\ & F_\epsilon & D_\epsilon & E_\epsilon & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & F_\epsilon & D_\epsilon & E_\epsilon \\ & & & & & F_\epsilon & D_\epsilon \end{bmatrix}_{M \times M},$$

where  $A_\epsilon = A + \tilde{A}_\epsilon$ ,  $B_\epsilon = -A + \tilde{B}_\epsilon$ ,  $C_\epsilon = A + \tilde{C}_\epsilon$ ,  $E_\epsilon = -A + \tilde{E}_\epsilon$ ,  $F_\epsilon = A + \tilde{E}_\epsilon$ , and

$$(3.5b) \quad A = \left( \frac{h_2}{12} \right) S,$$

$$(3.5c) \quad \tilde{A}_\epsilon = \left( \frac{(\xi_1 - 2\epsilon)h_2}{12h_1} \right) S + \epsilon \left( \frac{\xi_2 + 2h_1^2 - 9h_1\epsilon + 12\epsilon^2}{6h_1h_2} \right) U,$$

$$(3.5d) \quad \tilde{B}_\epsilon = \left( \frac{(\xi_3 + 2\epsilon)h_2}{12h_1} \right) S + \epsilon \left( \frac{\xi_4 - 6\epsilon^2 + h_1^2}{6h_1h_2} \right) U,$$

$$(3.5e) \quad \tilde{C}_\epsilon = - \left( \frac{(\xi_3 + 2\epsilon)h_2}{12h_1} \right) S + \epsilon \left( \frac{\xi_4 - 6\epsilon^2 + h_1^2}{6h_1h_2} \right) U,$$

$$(3.5f) \quad \tilde{E}_\epsilon = -\epsilon \left( \frac{h_2}{6h_1} \right) S + \epsilon \left( \frac{h_1}{6h_2} \right) U,$$

$$(3.5g) \quad D_\epsilon = \epsilon \left( \frac{h_2}{3h_1} \right) S + \epsilon \left( \frac{2h_1}{3h_2} \right) U,$$

where the  $\xi_i$  are all exponentially small as  $\epsilon \rightarrow 0$ :

$$(3.5h) \quad \xi_1 = (-2\epsilon - h_1) \exp \left( -\frac{2h_1}{\epsilon} \right) + 4\epsilon \exp \left( -\frac{h_1}{\epsilon} \right),$$

$$(3.5i) \quad \xi_2 = (9h_1\epsilon + 2h_1^2 + 12\epsilon^2) \exp \left( -\frac{2h_1}{\epsilon} \right) + (2h_1^2 - 24\epsilon^2) \exp \left( -\frac{h_1}{\epsilon} \right),$$

$$(3.5j) \quad \xi_3 = (-2\epsilon - h_1) \exp \left( -\frac{h_1}{\epsilon} \right),$$

$$(3.5k) \quad \xi_4 = (6h_1\epsilon + 6\epsilon^2 + 2h_1^2) \exp \left( -\frac{h_1}{\epsilon} \right).$$

The matrix  $\Gamma_\epsilon$  is block tridiagonal, and each block,  $A_\epsilon$  to  $F_\epsilon$ , is tridiagonal too. The blank blocks are  $(N-1) \times (N-1)$ - zero matrices, we will sometimes denote a zero matrix by 0 if it needs to be distinguished.

It is noteworthy that the matrices  $\tilde{A}_\epsilon$  to  $\tilde{E}_\epsilon$  converge to 0 when  $\epsilon \rightarrow 0$  and

$$(3.6) \quad A_\epsilon, C_\epsilon, F_\epsilon \rightarrow A, \quad B_\epsilon, E_\epsilon \rightarrow -A, \quad D_\epsilon \rightarrow 0.$$

The limit matrix, i.e.  $\Gamma_\epsilon$  when  $\epsilon = 0$ , is a simple structured block matrix ( $= A\Lambda_0$  as in (3.17) below) and, furthermore, its inverse matrix can be found explicitly, see Lemma 3.2 below.

Note that the evaluations of the entries of the matrix  $\Gamma_\epsilon$  were performed as follows:

$$(3.7a) \quad a_\epsilon(\phi_i \psi_j, \phi_k \psi_l) = \begin{cases} \epsilon B_{0,h_1}^2 L_{j,l,h_2}^1 + \epsilon B_{0,h_1}^1 L_{j,l,h_2}^2 & \text{if } i = 0, k = 0, \\ \epsilon B_{i,h_1}^2 L_{j,l,h_2}^1 + \epsilon B_{i,h_1}^1 L_{j,l,h_2}^2 + B_{i,h_1}^3 L_{j,l,h_2}^1 & \text{if } i \geq 1, k = 0, \\ \epsilon B_{k,h_1}^2 L_{j,l,h_2}^1 + \epsilon B_{k,h_1}^1 L_{j,l,h_2}^2 - B_{k,h_1}^3 L_{j,l,h_2}^1 & \text{if } i = 0, k \geq 1, \\ \epsilon L_{i,k,h_1}^2 L_{j,l,h_2}^1 + \epsilon L_{i,k,h_1}^1 L_{j,l,h_2}^2 - L_{i,k,h_1}^3 L_{j,l,h_2}^1 & \text{if } i \geq 1, k \geq 1, \end{cases}$$

where

$$(3.7b) \quad B_{k,h_1}^1 = \int_0^1 \phi_0 \phi_k dx = \begin{cases} \frac{1}{6h_1} \{ \xi_2 + 2h_1^2 - 9h_1\epsilon + 12\epsilon^2 \} & \text{if } k = 0, \\ \frac{1}{6h_1} \{ \xi_4 - 6\epsilon^2 + h_1^2 \} & \text{if } k = 1, \\ 0 & \text{if } k \geq 2, \end{cases}$$

$$(3.7c) \quad B_{k,h_1}^2 = \int_0^1 \frac{d\phi_0}{dx} \frac{d\phi_k}{dx} dx = \begin{cases} \frac{1}{2h_1\epsilon} \{ \xi_1 - 2\epsilon + h_1 \} & \text{if } k = 0, \\ 0 & \text{if } k \geq 1, \end{cases}$$

$$(3.7d) \quad B_{k,h_1}^3 = \int_0^1 \frac{d\phi_0}{dx} \phi_k dx = \begin{cases} \frac{1}{2h_1} \{ \xi_3 + 2\epsilon - h_1 \} & \text{if } k = 1, \\ 0 & \text{if } k = 0, \text{ or } k \geq 2. \end{cases}$$

Here  $\xi_1$  to  $\xi_4$  are as in (3.5h) - (3.5k), and

$$(3.7e) \quad L_{j,l,h_2}^1 = \int_0^1 \psi_j \psi_l dy = \begin{cases} \frac{2h_2}{3} & \text{if } l = j, \\ \frac{h_2}{6} & \text{if } l = j+1 \text{ or } l = j-1, \\ 0 & \text{otherwise,} \end{cases}$$

$$(3.7f) \quad L_{j,l,h_2}^2 = \int_0^1 \frac{d\psi_j}{dy} \frac{d\psi_l}{dy} dy = \begin{cases} \frac{2}{h_2} & \text{if } l = j, \\ -\frac{1}{h_2} & \text{if } l = j+1 \text{ or } l = j-1, \\ 0 & \text{otherwise,} \end{cases}$$

$$(3.7g) \quad L_{j,l,h_2}^3 = \int_0^1 \frac{d\psi_j}{dy} \psi_l dy = \begin{cases} -\frac{1}{2} & \text{if } l = j+1, \\ \frac{1}{2} & \text{if } l = j-1, \\ 0 & \text{otherwise.} \end{cases}$$

We now consider the two following cases, namely when  $\epsilon \leq \kappa h_2^2$  and when  $\kappa h_2^2 \leq \epsilon \leq \kappa h_2$ . The case  $\epsilon \leq \kappa h_2^2$ , which is presented in the next section, gives us some insights on why the new scheme (2.10) is stable and the classical scheme is not.

**3.1. Case  $\epsilon \leq \kappa h_2^2$ .** We will use Lemma 3.1 below which estimates the matrix norm of a (block) band matrix or dense matrix.

Let  $A_{ki}$  denote the  $(k+1, i+1)^{th}$  block in the block matrix  $\Lambda$ . We define its bandwidth  $w$  as follows:  $w = p+q-1$  if the entry blocks  $A_{ki} = 0$  whenever  $k+p \leq i$  or  $i+q \leq k$ .

LEMMA 3.1. *Let  $w$  be the bandwidth of a block matrix  $\Lambda$  with blocks  $\{A_{ki}\}$ . Then*

$$(3.8) \quad \|\Lambda\| \leq w \times \max_{k,i} \{\|A_{ki}\|\}.$$

PROOF. Let  $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_i, \dots, \mathbf{x}_{M-1})^T$ . We then easily verify that

$$(3.9) \quad \|\Lambda \mathbf{x}\|_2^2 = \sum_{k=0}^{M-1} \left\| \sum_{i=k-q+1}^{k+p-1} A_{ki} \mathbf{x}_i \right\|_2^2 \leq \max_{k,i} \{\|A_{ki}\|^2\} \sum_{k=0}^{M-1} \left[ \sum_{i=k-q+1}^{k+p-1} \|\mathbf{x}_i\|_2 \right]^2,$$



where  $p, q$  are as above. By the Cauchy-Schwarz inequality

$$(3.10) \quad \left[ \sum_{i=k-q+1}^{k+p-1} \|\mathbf{x}_i\|_2 \right]^2 \leq (p+q-1) \sum_{i=k-q+1}^{k+p-1} \|\mathbf{x}_i\|_2^2,$$

and hence, permuting the summation

$$(3.11) \quad \sum_{k=0}^{M-1} \left[ \sum_{i=k-q+1}^{k+p-1} \|\mathbf{x}_i\|_2 \right]^2 \leq (p+q-1)^2 \sum_{i=0}^{M-1} \|\mathbf{x}_i\|_2^2.$$

Therefore, from (3.9) and (3.11),

$$(3.12) \quad \|\Lambda\| = \max_{\|\mathbf{x}\|_2=1} \|\Lambda \mathbf{x}\|_2 \leq (p+q-1) \max_{k,i} \{\|A_{ki}\|\};$$

the lemma follows.  $\square$

REMARK 3.1. The norm of each  $\|A_{ki}\|$  can be estimated as in Lemma 3.1. More precisely, if  $A_{ki}$  is a band matrix with a bandwidth  $\bar{w}$ , then

$$(3.13) \quad \|A_{ki}\| \leq \bar{w} \times \max_{l,j} \{|a_{lj}^{ki}|\},$$

where  $a_{lj}^{ki}$  is the  $(l, j)^{th}$  entry of  $A_{ki}$ .

In particular, if the matrix  $\Lambda$  is of size  $M \times M$  and its bandwidth  $w$  depends on  $M = 1/h_1$ , e.g. a matrix with no zero entries, we easily see that since  $w \leq 2M$ ,

$$(3.14) \quad \|\Lambda\| \leq \frac{\kappa}{h_1} \times \max_{k,i} \{\|A_{ki}\|\}.$$

Furthermore, if each block  $A_{ki}$  is of size  $(N-1) \times (N-1)$  and its bandwidth  $\bar{w}$  depends on  $N = 1/h_2$ , since  $\bar{w} \leq 2(N-1)$ , we infer from (3.13), (3.14) that

$$(3.15) \quad \|\Lambda\| \leq w \times \bar{w} \times \max_{l,j} \{|a_{lj}^{ki}|\} \leq \frac{\kappa}{h_1 h_2} \times \max_{l,j,k,i} \{|a_{lj}^{ki}|\}.$$

Thanks to (3.5), multiplying equation (3.2) by  $A^{-1}$ , we write this equation as follows:

$$(3.16) \quad A^{-1} \Gamma_\epsilon \mathbf{a} = (\Lambda_0 + \Lambda_\epsilon) \mathbf{a} = \tilde{\mathbf{b}},$$

where

$$(3.17) \quad \Lambda_0 = \begin{bmatrix} I & -I & & & \\ I & 0 & -I & & \\ & I & 0 & -I & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & I & 0 & -I \\ & & & & I & 0 & \end{bmatrix}_{M \times M},$$

$$(3.18) \quad \Lambda_\epsilon = \begin{bmatrix} A^{-1}\tilde{A}_\epsilon & A^{-1}\tilde{B}_\epsilon & & & \\ A^{-1}\tilde{C}_\epsilon & A^{-1}D_\epsilon & A^{-1}\tilde{E}_\epsilon & & \\ & A^{-1}\tilde{E}_\epsilon & A^{-1}D_\epsilon & A^{-1}\tilde{E}_\epsilon & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & A^{-1}\tilde{E}_\epsilon & A^{-1}D_\epsilon & A^{-1}\tilde{E}_\epsilon \\ & & & & A^{-1}\tilde{E}_\epsilon & A^{-1}D_\epsilon & \end{bmatrix}_{M \times M},$$

and

$$(3.19) \quad \tilde{\mathbf{b}} = (A^{-1}\mathbf{b}_0, A^{-1}\mathbf{b}_1, \dots, A^{-1}\mathbf{b}_{M-1})^T.$$

We now assume that the ratio  $h_2/h_1$  is bounded:

$$(3.20) \quad (\text{H2}) \quad h_2 \leq \kappa h_1.$$

We then notice that

$$(3.21) \quad \|\tilde{A}_\epsilon\| \leq \kappa \frac{\epsilon h_1}{h_2};$$

indeed, from the entries of the matrix  $\tilde{A}_\epsilon$  shown in (3.5c), using the hypotheses (H0)-(2.6), (H1)-(2.7) and (H2)-(3.20), after some elementary calculations, we find that the entries of the block  $\tilde{A}_\epsilon$  are majorized by  $\kappa \epsilon h_1/h_2$ . Furthermore, since  $\tilde{A}_\epsilon$  is banded, (3.21) follows from estimate (3.13). Similarly, from (3.5d)-(3.5g) we find

$$(3.22) \quad \|\tilde{B}_\epsilon\|, \|\tilde{C}_\epsilon\|, \|\tilde{E}_\epsilon\|, \|D_\epsilon\| \leq \kappa \frac{\epsilon h_1}{h_2}.$$

From (3.3) and (3.5b), we find

$$(3.23) \quad \|A^{-1}\| \leq \frac{\kappa}{h_2} \|S^{-1}\| \leq \frac{\kappa}{h_2}.$$

Using estimates (3.21), we then find

$$(3.24) \quad \|A^{-1}\tilde{A}_\epsilon\| \leq \|A^{-1}\| \|\tilde{A}_\epsilon\| \leq \kappa \frac{\epsilon h_1}{h_2^2},$$

and similarly, by (3.22), we find

$$(3.25) \quad \|A^{-1}\tilde{B}_\epsilon\|, \|A^{-1}\tilde{C}_\epsilon\|, \|A^{-1}\tilde{E}_\epsilon\|, \|A^{-1}D_\epsilon\| \leq \kappa \frac{\epsilon h_1}{h_2^2}.$$

Hence, from Lemma 3.1 and (3.18), since  $\Lambda_\epsilon$  is banded, we easily find

$$(3.26) \quad \|\Lambda_\epsilon\| \leq \kappa \frac{\epsilon h_1}{h_2^2},$$

and it is not hard to see that

$$(3.27) \quad \|\tilde{\mathbf{b}}\|_2 \leq \|A^{-1}\| \|\mathbf{b}\|_2 \leq \frac{\kappa}{h_2} \|\mathbf{b}\|_2.$$

Taking the norm of each side of equation (3.16), we find

$$(3.28) \quad \|(\Lambda_0 + \Lambda_\epsilon)\mathbf{a}\|_2 = \|\tilde{\mathbf{b}}\|_2 \leq \frac{\kappa}{h_2} \|\mathbf{b}\|_2.$$

We are now able to estimate the norm  $\|\mathbf{a}\|_2$  as follows. Firstly,

$$(3.29) \quad \begin{aligned} \|\Lambda_0\mathbf{a}\|_2 &\leq \|(\Lambda_0 + \Lambda_\epsilon)\mathbf{a}\|_2 + \|-\Lambda_\epsilon\mathbf{a}\|_2 \\ &\leq \frac{\kappa}{h_2} \|\mathbf{b}\|_2 + \|\Lambda_\epsilon\| \|\Lambda_0^{-1}\| \|\Lambda_0\mathbf{a}\|_2 \\ &\leq (\text{by estimate (3.26) and Lemma 3.2 below}) \\ &\leq \frac{\kappa}{h_2} \|\mathbf{b}\|_2 + \kappa_1 \frac{\epsilon}{h_2^2} \|\Lambda_0\mathbf{a}\|_2. \end{aligned}$$

Note here that we named the constant  $\kappa_1$  and we now assume that

$$(3.30) \quad (\text{H3}) \quad \kappa_1 \frac{\epsilon}{h_2^2} \leq \frac{1}{2}, \left( \text{or } \epsilon \leq \frac{h_2^2}{2\kappa_1} \right).$$

We then deduce from (3.29) that

$$(3.31) \quad \|\Lambda_0\mathbf{a}\|_2 \leq \frac{\kappa}{h_2} \|\mathbf{b}\|_2,$$

and hence

$$(3.32) \quad \|\mathbf{a}\|_2 \leq \|\Lambda_0^{-1}\| \|\Lambda_0 \mathbf{a}\|_2 \leq \frac{\kappa}{h_1 h_2} \|\mathbf{b}\|_2.$$

We now justify the estimate of the norm of  $\Lambda_0^{-1}$  and derive the relations between  $\|\mathbf{a}\|_2$  and  $|u_N|_{L^2}$ , and between  $\|\mathbf{b}\|_2$  and  $|f|_{L^2}$  in the subsequent lemmas.

LEMMA 3.2. *The inverse  $\Lambda_0^{-1}$  of  $\Lambda_0$  is given by formulas (3.34) below and we have:*

$$(3.33) \quad \|\Lambda_0^{-1}\| \leq \frac{\kappa}{h_1}.$$

PROOF. The inverse matrix  $\Lambda_0^{-1}$  can be found recursively as follows. Set

$$(3.34a) \quad \Xi(1) = I, \quad \Xi(2) = \begin{bmatrix} 0 & I \\ -I & I \end{bmatrix}.$$

Then we claim that

$$(3.34b) \quad \Lambda_0^{-1} = \Xi(M) = \begin{bmatrix} \Xi(M-2) & \Xi_1 \\ & \Xi_2 \\ \Xi_3 & \Xi_4 \end{bmatrix},$$

where

$$(3.34c) \quad \Xi_1 = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ I & I & I & \cdot & I & I & I \end{bmatrix}_{2 \times (M-2)}^T, \quad \Xi_2 = \begin{bmatrix} 0 & I \\ -I & I \end{bmatrix}_{2 \times 2},$$

and for  $M = 2m$  and  $M = 2m + 1$ , respectively,

$$(3.34d) \quad \begin{aligned} \Xi_3 &= \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ -I & I & -I & \cdot & I & -I & I \end{bmatrix}_{2 \times (2m-2)}, \\ \Xi_4 &= \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ I & -I & I & \cdot & I & -I & I \end{bmatrix}_{2 \times (2m-1)}; \end{aligned}$$

note that the minus sign is alternating<sup>1</sup>. Then since the entries of  $\Lambda_0^{-1}$  are 0 or  $\pm I$  blocks, the estimate (3.33) follows from (3.14).

To prove that the matrix  $\Xi(M)$  in (3.34) is indeed the inverse of the matrix  $\Lambda_0$ , we first consider the case  $M = 2m$  and use an induction on  $m$  (the case  $M = 2m + 1$  can be done similarly). For  $m = 1$ , we easily verify that  $\Xi(2)$  is the inverse of  $\Lambda_0$ . Suppose that for  $M = 2m$ ,  $m \geq 1$ ,  $\Xi(M)$  is the inverse of  $\Lambda_0$ . We then verify that for  $M = 2(m + 1)$ ,  $\Xi(M)$  is the inverse of  $\Lambda_0$ : indeed we rewrite  $\Lambda_0$  in (3.17) in the form:

$$(3.35) \quad \Lambda_0 = \Lambda(M) = \begin{bmatrix} \Lambda(M-2) & \Lambda_1 \\ & \Lambda_2 \\ \Lambda_3 & \end{bmatrix}_{M \times M},$$

where

$$(3.36) \quad \Lambda(M-2) = \begin{bmatrix} I & -I & & & \\ I & 0 & -I & & \\ & I & 0 & \cdot & \\ & & \cdot & \cdot & -I \\ & & & I & 0 \end{bmatrix}_{(M-2) \times (M-2)},$$

---

<sup>1</sup>For example, for  $m = 4$ ,

$$\Lambda_0^{-1} = \begin{bmatrix} 0 & I & 0 & I & 0 & I & 0 & I \\ -I & I & 0 & I & 0 & I & 0 & I \\ 0 & 0 & 0 & I & 0 & I & 0 & I \\ -I & I & -I & I & 0 & I & 0 & I \\ 0 & 0 & 0 & 0 & 0 & I & 0 & I \\ -I & I & -I & I & -I & I & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ -I & I & -I & I & -I & I & -I & I \end{bmatrix}_{8 \times 8}, \quad \begin{bmatrix} I & 0 & I & 0 & I & 0 & I & 0 & I \\ 0 & 0 & I & 0 & I & 0 & I & 0 & I \\ I & -I & I & 0 & I & 0 & I & 0 & I \\ 0 & 0 & 0 & 0 & I & 0 & I & 0 & I \\ I & -I & I & -I & I & 0 & I & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & I \\ I & -I & I & -I & I & -I & I & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ I & -I & I & -I & I & -I & I & -I & I \end{bmatrix}_{9 \times 9}.$$

and

$$(3.37) \quad \Lambda_1 = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & -I \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{2 \times (M-2)}^T,$$

$$(3.38) \quad \Lambda_2 = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}_{2 \times 2}, \quad \Lambda_3 = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 & I \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{2 \times (M-2)}.$$

Then by explicit calculations, we find that

$$(3.39) \quad \Lambda_0 \Xi(M) = \begin{bmatrix} \Lambda(M-2)\Xi(M-2) + \Lambda_1\Xi_3 & \Lambda(M-2)\Xi_1 + \Lambda_1\Xi_2 \\ \Lambda_3\Xi(M-2) + \Lambda_2\Xi_3 & \Lambda_3\Xi_1 + \Lambda_2\Xi_2 \end{bmatrix}_{M \times M} = \tilde{I}.$$

Indeed, by our assumption, we find that  $\Lambda(M-2)\Xi(M-2) = \tilde{I}_{(M-2) \times (M-2)}$  and the other entries are computed explicitly.  $\square$

LEMMA 3.3. *For  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $u_N$  as in (3.1)-(3.2), the following relations hold:*

$$(3.40a) \quad |u_N|_{L^2} \leq \kappa(h_1 h_2)^{1/2} \|\mathbf{a}\|_2,$$

$$(3.40b) \quad \|\mathbf{b}\|_2 \leq \kappa(h_1 h_2)^{1/2} |f|_{L^2}.$$

PROOF. We easily verify that, by the compact supports of the elements  $\phi_i$  and  $\psi_j$ ,

$$(3.41) \quad \begin{aligned} |u_N|_{L^2}^2 &= \int_{\Omega} \left( \sum_{i=0}^{M-1} \sum_{j=1}^{N-1} a_{ij} \phi_i \psi_j \right)^2 d\Omega \\ &= \sum_{i=0}^{M-1} \sum_{j=1}^{N-1} \sum_{\substack{k \in \{i-1, i, i+1\}, \\ l \in \{j-1, j, j+1\}}} a_{ij} a_{kl} \int_0^1 \phi_i \phi_k dx \int_0^1 \psi_j \psi_l dy \\ &\leq \kappa h_1 h_2 \sum_{i=0}^{M-1} \sum_{j=1}^{N-1} a_{ij}^2, \end{aligned}$$

and this is exactly (3.40a). To verify (3.40b), we notice that

$$(3.42) \quad \mathbf{b} = \begin{bmatrix} \int_{\Omega_{01}} f \phi_0 \psi_1 d\Omega \\ \int_{\Omega_{02}} f \phi_0 \psi_2 d\Omega \\ \cdot \\ \int_{\Omega_{kl}} f \phi_k \psi_l d\Omega \\ \cdot \\ \int_{\Omega_{M-1,N-1}} f \phi_{M-1} \psi_{N-1} d\Omega \end{bmatrix},$$

where the  $\Omega_{kl}$  are the compact supports of the elements  $\phi_k \psi_l$ , more precisely,

$$\Omega_{0l} = (0, h_1) \times ((l-1)h_2, (l+1)h_2) \text{ for } k = 0,$$

$$\Omega_{kl} = ((k-1)h_1, (k+1)h_1) \times ((l-1)h_2, (l+1)h_2) \text{ for } k \geq 1.$$

We then find by the Cauchy-Schwarz inequality that

$$(3.43) \quad \begin{aligned} \|\mathbf{b}\|_2^2 &= \sum_{k=0}^{M-1} \sum_{l=1}^{N-1} \left( \int_{\Omega_{kl}} f \phi_k \psi_l d\Omega \right)^2 \\ &\leq \sum_{k=0}^{M-1} \sum_{l=1}^{N-1} \int_{\Omega_{kl}} f^2 d\Omega \int_{\Omega_{kl}} (\phi_k \psi_l)^2 d\Omega \\ &\leq (\text{by (3.7e)}) \leq \kappa h_1 h_2 \int_{\Omega} f^2 d\Omega; \end{aligned}$$

hence, (3.40b) follows.  $\square$

Using the estimate (3.32) and Lemma 3.3, we can directly deduce the following theorem.

**THEOREM 3.1.** *Assume that (H0)-(H3) hold, that is [(2.6), (2.7), (3.20), (3.30)]. Let  $u_N$  be the solution of problem (2.10). Then for any data  $f = f(x, y) \in L^2(\Omega)$  (not necessarily satisfying (1.2)), there exists a constant  $\kappa > 0$  independent of  $\epsilon$ ,  $h_1$ , and  $h_2$  such that*

$$(3.44) \quad |u_N|_{L^2(\Omega)} \leq \kappa |f|_{L^2(\Omega)}.$$

REMARK 3.2. For a classical scheme not using a BLE, the stiffness matrix  $\Gamma_\epsilon$  is as in (3.5a), after deleting the first row and the first column of the matrix in (3.5a). Hence, since from (3.6), as  $\epsilon \rightarrow 0$ ,  $D_\epsilon \rightarrow 0$ ,  $E_\epsilon \rightarrow -A$  and  $F_\epsilon \rightarrow A$ , it is obvious that the system tends to a singular system. On the other hand, for the new scheme (2.10), the entries  $A_\epsilon$ ,  $B_\epsilon$ , and  $C_\epsilon$  in  $\Gamma_\epsilon$  stabilize our system as we have seen in this section, i.e. the BLE  $\phi_0$  absorbs the singularity due to the small  $\epsilon$  of the linear system (3.2).

**3.2. Case  $\kappa h_2^2 \leq \epsilon \leq \kappa h_2$ .** If we assume that

$$(3.45) \quad (\text{H4}) \quad \kappa_2 h_2^2 \leq \epsilon \leq \kappa h_2, \quad \kappa_2 = 1/(2\kappa_1),$$

we easily see that we cannot derive (3.31) from (3.29). We will need some more delicate analysis which we introduce in this section; in particular we need to investigate more carefully the BLE  $\phi_0$  introduced in (2.3).

To obtain the  $L^2$ -stability in this range of values of  $\epsilon$ , we will utilize quasi-uniform elements, namely we assume

$$(3.46) \quad (\text{H5}) \quad \sqrt{2}h_1 \leq h_2 \leq \kappa h_1;$$

the  $\sqrt{2}$  will be justified later; note that (H5)-(3.46) implies (H2)-(3.20).

We first derive in Lemma 3.4 below, a Poincaré-like inequality for any  $v_N^{BL}$ . For  $v_N \in V_N$ , we write  $v_N = v_N^{BL} + v_N^{LI}$ , where

$$(3.47) \quad v_N^{BL} = \sum_{j=1}^{N-1} a_{0j} \phi_0 \psi_j, \quad v_N^{LI} = \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} a_{ij} \phi_i \psi_j,$$

and we have:

LEMMA 3.4. *There exists a positive constant  $\kappa$  independent of  $\epsilon$ ,  $h_1$ , and  $h_2$  such that, for any  $v_N \in V_N$ ,*

$$(3.48) \quad |v_N^{BL}|_{L^2(\Omega)} \leq \kappa \epsilon^{1/2} h_1^{1/2} |v_N|_{H^1(\Omega)}.$$



PROOF. Firstly, we notice that  $Mh_1 = Nh_2 = 1$ , and due to the boundary conditions,

$$(3.49) \quad a_{M,l} = a_{k,0} = a_{k,N} = 0,$$

and by explicit calculations,

$$(3.50) \quad \int_{(l-1)h_2}^{lh_2} \psi_l^2 dy = \int_{(l-1)h_2}^{lh_2} \psi_{l-1}^2 dy = 2 \int_{(l-1)h_2}^{lh_2} \psi_l \psi_{l-1} dy = \frac{h_2}{3}.$$

We now derive the estimates (3.48) as follows. Taking into consideration the supports of the elements  $\phi_k$  and  $\psi_l$ , we see that

$$(3.51) \quad \begin{aligned} \int_{\Omega} \left( \frac{\partial v_N}{\partial x} \right)^2 d\Omega &= \sum_{k=1}^M \sum_{l=1}^N \int_{\tilde{\Omega}_{kl}} \left( \frac{\partial v_N}{\partial x} \right)^2 d\Omega \\ &= \sum_{k=1}^M \sum_{l=1}^N \int_{\tilde{\Omega}_{kl}} \left( \psi_l \frac{d\tilde{\phi}_{k,l}}{dx} + \psi_{l-1} \frac{d\tilde{\phi}_{k,l-1}}{dx} \right)^2 d\Omega, \end{aligned}$$

where

$$(3.52) \quad \tilde{\Omega}_{kl} = ((k-1)h_1, kh_1) \times ((l-1)h_2, lh_2),$$

$$(3.53) \quad \tilde{\phi}_{k,j} = a_{k,j}\phi_k + a_{k-1,j}\phi_{k-1}, \quad j = l, l-1.$$

We find from (3.50) that

$$(3.54) \quad \begin{aligned} &\int_{\tilde{\Omega}_{kl}} \left( \psi_l \frac{d\tilde{\phi}_{k,l}}{dx} + \psi_{l-1} \frac{d\tilde{\phi}_{k,l-1}}{dx} \right)^2 d\Omega \\ &= \frac{h_2}{3} \int_{(k-1)h_1}^{kh_1} \left\{ \left( \frac{d\tilde{\phi}_{k,l}}{dx} \right)^2 + \frac{d\tilde{\phi}_{k,l}}{dx} \frac{d\tilde{\phi}_{k,l-1}}{dx} + \left( \frac{d\tilde{\phi}_{k,l-1}}{dx} \right)^2 \right\} dx \\ &\geq I_{kl} + I_{k,l-1}, \quad (\text{using } a^2 + ab + b^2 \geq (a^2 + b^2)/2), \end{aligned}$$

where

$$(3.55) \quad I_{kj} = \frac{h_2}{6} \int_{(k-1)h_1}^{kh_1} \left( \frac{d\tilde{\phi}_{k,l}}{dx} \right)^2 dx, \quad j = l, l-1.$$

For  $k = 1$ , since  $\int_0^{h_1} (d\phi_0/dx) dx = \phi_0(h_1) - \phi_0(0) = 0$ ,

$$(3.56) \quad \int_0^{h_1} \left( \frac{d\tilde{\phi}_{1,l}}{dx} \right)^2 dx = \frac{a_{1,l}^2}{h_1} + a_{0,l}^2 \int_0^{h_1} \left( \frac{d\phi_0}{dx} \right)^2 dx.$$

By further calculations, we find in the expression (3.7c),

$$(3.57) \quad \begin{aligned} \xi_1 &= \left\{ 4\epsilon - (2\epsilon + h_1) \exp\left(-\frac{h_1}{\epsilon}\right) \right\} \exp\left(-\frac{h_1}{\epsilon}\right) \\ &\geq (\text{by (H0)-(2.6)}) \geq (4 - 2\epsilon - h_1)\epsilon \exp\left(-\frac{h_1}{\epsilon}\right) \geq 0; \end{aligned}$$

this implies that by (H1)-(2.7)

$$(3.58) \quad \int_0^{h_1} \left( \frac{d\phi_0}{dx} \right)^2 dx = B_{0,h_1}^2 \geq \frac{-2\epsilon + h_1}{2h_1\epsilon} \geq \frac{1}{4\epsilon}.$$

Hence, from (3.55), (3.56) and (3.58),

$$(3.59) \quad I_{1,l} = \frac{h_2}{6} \int_0^{h_1} \left( \frac{d\tilde{\phi}_{1,l}}{dx} \right)^2 dx \geq a_{0,l}^2 \frac{h_2}{24\epsilon}.$$

For  $k \geq 2$ , we observe that

$$(3.60) \quad I_{k,l} = \frac{h_2}{6} \int_{(k-1)h_1}^{kh_1} \left( \frac{d\tilde{\phi}_{k,l}}{dx} \right)^2 dx = \frac{h_2(a_{k,l} - a_{k-1,l})^2}{6h_1}.$$

Now using (3.51), (3.54), (3.59) and the positivity of the  $I_{k,l}$ , we find that

$$(3.61) \quad \int_{\Omega} \left( \frac{\partial v_N}{\partial x} \right)^2 d\Omega \geq \sum_{k=1}^M \sum_{l=1}^N \{ I_{kl} + I_{k,l-1} \} \geq \sum_{l=1}^N I_{1,l} \geq \frac{h_2}{24\epsilon} \sum_{l=1}^N a_{0,l}^2,$$

and thus

$$(3.62) \quad h_2 \sum_{j=1}^{N-1} a_{0,j}^2 \leq \kappa \epsilon |v_N|_{H^1}^2.$$

Thanks to (H0)-(2.6), (H1)-(2.7), we easily verify that  $\int_0^1 \phi_0^2 dx \leq \kappa h_1$  from (3.7b) and we thus have

$$(3.63) \quad \begin{aligned} |v_N^{BL}|_{L^2}^2 &= \int_{\Omega} \left( \sum_{j=1}^{N-1} a_{0j} \phi_0 \psi_j \right)^2 d\Omega \\ &= \sum_{j=1}^{N-1} \sum_{l \in \{j-1, j, j+1\}} a_{0j} a_{0l} \int_0^1 \psi_j \psi_l dy \int_0^1 \phi_0^2 dx \\ &\leq \kappa h_1 h_2 \sum_{j=1}^{N-1} a_{0j}^2 \leq (\text{by (3.62)}) \leq \kappa h_1 \epsilon |v_N|_{H^1}^2, \end{aligned}$$

and the lemma follows.  $\square$

REMARK 3.3. Later on we will use the following inequality: from (3.60) and (3.61) with  $v_N = u_N$ , where  $u_N$  is the solution of equation (2.10), we can write

$$(3.64) \quad \frac{h_2}{6h_1} \sum_{k=2}^M \|\mathbf{a}_{k-1} - \mathbf{a}_k\|_2^2 \leq \sum_{k=2}^M \sum_{l=1}^N I_{k,l} \leq \int_{\Omega} \left( \frac{\partial u_N}{\partial x} \right)^2 d\Omega.$$

We now estimate  $\|\mathbf{a}\|_2$  to obtain the upper bound of  $|u_N|_{L^2}$  as indicated in Lemma 3.3. For that purpose we write the system (3.2) in the more explicit form:

$$(3.65a) \quad A_{\epsilon} \mathbf{a}_0 + B_{\epsilon} \mathbf{a}_1 = \mathbf{b}_0,$$

$$(3.65b) \quad C_{\epsilon} \mathbf{a}_0 + D_{\epsilon} \mathbf{a}_1 + E_{\epsilon} \mathbf{a}_2 = \mathbf{b}_1,$$

$$(3.65c) \quad F_{\epsilon} \mathbf{a}_{i-2} + D_{\epsilon} \mathbf{a}_{i-1} + E_{\epsilon} \mathbf{a}_i = \mathbf{b}_{i-1}, \text{ for } i = 3, \dots, M-1,$$

$$(3.65d) \quad F_{\epsilon} \mathbf{a}_{M-2} + D_{\epsilon} \mathbf{a}_{M-1} = \mathbf{b}_{M-1}.$$

Using (3.5) and setting  $\mathbf{a}_M = 0$ , we rewrite (3.65c) and (3.65d): for  $k = 3, \dots, M$ ,

$$(3.66) \quad (A + \tilde{E}_{\epsilon}) \mathbf{a}_{k-2} + D_{\epsilon} \mathbf{a}_{k-1} + (-A + \tilde{E}_{\epsilon}) \mathbf{a}_k = \mathbf{b}_{k-1}.$$

Taking the inner product of (3.66) with  $\mathbf{a}_{k-1}$ , using the symmetry of  $A$  and  $\tilde{E}_{\epsilon}$ , and summing over  $k = i, \dots, M$ ,  $i \geq 3$ , we find after some elementary calculations:

$$(3.67a) \quad \langle (A + \tilde{E}_{\epsilon}) \mathbf{a}_{i-2}, \mathbf{a}_{i-1} \rangle + J = \sum_{k=i}^M \langle \mathbf{b}_{k-1}, \mathbf{a}_{k-1} \rangle,$$

where

$$(3.67b) \quad J = \sum_{k=i}^{M-1} \langle 2\tilde{E}_{\epsilon} \mathbf{a}_{k-1}, \mathbf{a}_k \rangle + \sum_{k=i}^M \langle D_{\epsilon} \mathbf{a}_{k-1}, \mathbf{a}_{k-1} \rangle.$$

We then claim that  $J \geq 0$ . We firstly notice that  $2\tilde{E}_{\epsilon} + D_{\epsilon} = \epsilon h_1/h_2 U$ , and thanks to the Gershgorin circle theorem, we find that the eigenvalues of  $U$  are nonnegative.

Hence,  $\langle U \mathbf{a}_{k-1}, \mathbf{a}_{k-1} \rangle \geq 0$ , and

$$(3.68) \quad J \geq -2 \sum_{k=i}^{M-1} \langle G \mathbf{a}_{k-1}, \mathbf{a}_k \rangle + 2 \sum_{k=i}^M \langle G \mathbf{a}_{k-1}, \mathbf{a}_{k-1} \rangle,$$

where  $G = -\tilde{E}_{\epsilon}$ . The quasi-uniform mesh hypothesis shows that  $G$  is positive semi-definite. Indeed, from (3.5f), we have  $G = -\tilde{E}_{\epsilon} = \epsilon (h_2/6h_1) S - \epsilon (h_1/6h_2) U$ . Then

its Gershgorin discs belong to

$$(3.69) \quad \mathcal{G} = \left\{ z \in \mathbb{C}; \left| z - \frac{\epsilon}{3} \left( \frac{2h_2}{h_1} - \frac{h_1}{h_2} \right) \right| \leq \frac{\epsilon}{3} \left( \frac{h_2}{h_1} + \frac{h_1}{h_2} \right) \right\}.$$

From (H5)-(3.46) we find that  $h_2/h_1 - 2h_1/h_2 \geq 0$  which guarantees that the Gershgorin discs belong to  $\mathbb{C}$  with nonnegative real parts. Since  $G$  is symmetric and thus its eigenvalues are real numbers, all eigenvalues of  $G$  are nonnegative. By the spectral property of  $G$ , we then write  $\langle G\xi, \eta \rangle = \langle G^{1/2}\xi, G^{1/2}\eta \rangle$  and hence we rewrite (3.68):

$$(3.70) \quad J \geq \sum_{k=i}^{M-1} \|G^{1/2}\mathbf{a}_{k-1} - G^{1/2}\mathbf{a}_k\|_2^2 + \langle G\mathbf{a}_{i-1}, \mathbf{a}_{i-1} \rangle + \langle G\mathbf{a}_{M-1}, \mathbf{a}_{M-1} \rangle \geq 0.$$

Hence from (3.67) we find that

$$(3.71) \quad \langle (A + \tilde{E}_\epsilon)\mathbf{a}_{i-2}, \mathbf{a}_{i-1} \rangle \leq \sum_{k=i}^M \langle \mathbf{b}_{k-1}, \mathbf{a}_{k-1} \rangle.$$

Since  $\langle U\xi, \xi \rangle \geq 0$  and  $\langle S\xi, \xi \rangle \geq 2\|\xi\|_2^2$ , we find that by (H1)-(2.7)

$$(3.72) \quad \langle (A + \tilde{E}_\epsilon)\xi, \xi \rangle = \frac{h_2}{12} \left( 1 - \frac{2\epsilon}{h_1} \right) \langle S\xi, \xi \rangle + \epsilon \frac{h_1}{6h_1} \langle U\xi, \xi \rangle \geq \frac{h_2}{12} \|\xi\|_2^2,$$

and from (3.71) and the fact that  $\|A + \tilde{E}_\epsilon\|_2 \leq \kappa h_2$  we find

$$(3.73) \quad \begin{aligned} \frac{h_2}{12} \|\mathbf{a}_{i-2}\|_2^2 &\leq \langle (A + \tilde{E}_\epsilon)\mathbf{a}_{i-2}, \mathbf{a}_{i-2} \rangle \\ &\leq \sum_{k=i}^M \langle \mathbf{b}_{k-1}, \mathbf{a}_{k-1} \rangle + \langle (A + \tilde{E}_\epsilon)\mathbf{a}_{i-2}, \mathbf{a}_{i-2} - \mathbf{a}_{i-1} \rangle \\ &\leq \|\mathbf{a}\|_2 \|\mathbf{b}\|_2 + \frac{h_2}{24} \|\mathbf{a}_{i-2}\|_2^2 + \kappa h_2 \|\mathbf{a}_{i-2} - \mathbf{a}_{i-1}\|_2^2. \end{aligned}$$

Hence, summing (3.73) over  $i = 3$  to  $M + 1$  and multiplying by  $24h_1$ , we find that

$$(3.74) \quad h_1 h_2 \sum_{i=3}^{M+1} \|\mathbf{a}_{i-2}\|_2^2 \leq \kappa h_1 M \|\mathbf{a}\|_2 \|\mathbf{b}\|_2 + \kappa h_1 h_2 \sum_{i=3}^{M+1} \|\mathbf{a}_{i-2} - \mathbf{a}_{i-1}\|_2^2.$$

Thanks to (3.62), adding to (3.74)  $h_1 h_2 \|\mathbf{a}_0\|_2^2 \leq \kappa h_1 \epsilon |u_N|_{H^1}^2$ , and since  $|u_N|_{H^1}^2 \leq \epsilon^{-1} |f|_{L^2} |u_N|_{L^2}$  by letting  $v = u_N$  in (2.10b), we find that

$$\begin{aligned}
 (3.75) \quad h_1 h_2 \|\mathbf{a}\|_2^2 &\leq \kappa \|\mathbf{a}\|_2 \|\mathbf{b}\|_2 + \kappa h_1 \epsilon |u_N|_{H^1}^2 + \kappa h_1 h_2 \sum_{i=3}^{M+1} \|\mathbf{a}_{i-2} - \mathbf{a}_{i-1}\|_2^2 \\
 &\leq (\text{by (3.64)}) \leq \kappa \|\mathbf{a}\|_2 \|\mathbf{b}\|_2 + \kappa (h_1^2 + h_1 \epsilon) |u_N|_{H^1}^2 \\
 &\leq (\text{by (3.45), (3.46)}) \leq \kappa \|\mathbf{a}\|_2 \|\mathbf{b}\|_2 + \kappa |f|_{L^2} |u_N|_{L^2}.
 \end{aligned}$$

Hence from Lemma 3.3 valid in all cases:

$$|u_N|_{L^2}^2 \leq h_1 h_2 \|\mathbf{a}\|_2^2 \leq \kappa h_1^{-1} h_2^{-1} \|\mathbf{b}\|_2^2 + \kappa |f|_{L^2} |u_N|_{L^2} \leq \kappa |f|_{L^2}^2 + \kappa |f|_{L^2} |u_N|_{L^2}.$$

By the Cauchy-Schwarz inequality, we thus deduce the following theorem.

**THEOREM 3.2.** *Assume that the hypotheses (H0)-(H1), (H4)-(H5) hold, that is [(2.6), (2.7), (3.45), (3.46)]. Let  $u_N$  be the solution of problem (2.10). Then for any data  $f = f(x, y) \in L^2(\Omega)$  (not necessarily satisfying (1.2)), there exists a constant  $\kappa > 0$  independent of  $\epsilon$ ,  $h_1$ , and  $h_2$  such that*

$$(3.76) \quad |u_N|_{L^2(\Omega)} \leq \kappa |f|_{L^2(\Omega)}.$$

**REMARK 3.4.** For the problem (1.1a) with different boundary conditions, e.g.  $u = 0$  at  $x = 0, 1$  and  $\partial u / \partial y = 0$  at  $y = 0, 1$ , or  $u = 0$  at  $x = 0, 1$  and  $u(x, y) = u(x, y + 1)$ , which lead to a slight change of each block  $A_\epsilon$  to  $F_\epsilon$ , we can similarly verify Theorem 3.1 - 3.2.

#### 4. $H^1$ - and $L^2$ - Approximation Errors

The following Theorem 4.1 - 4.2 give the  $H^1$  and  $L^2$ - behavior of the convergence errors for the approximate solutions.

**THEOREM 4.1** ( $H^1$ - error). *Assume only that (H0)-(2.6), (H1)-(2.7) hold. Let  $u = u^\epsilon$  be the exact solution of (1.3), and  $u_N$  the solution of (2.10), and let  $f$  be*

smooth on  $\bar{\Omega}$  satisfying (1.2). Then

$$(4.1) \quad |u - u_N|_{H^1(\Omega)} \leq \kappa(\bar{h} + \bar{h}^2 \epsilon^{-1}).$$

PROOF. Subtracting (2.2) from (1.3), we find

$$(4.2) \quad a_\epsilon(u - u_N, v) = 0 \quad \text{for all } v \in V_N,$$

and thus for an interpolant  $\Pi u \in V_N$ ,  $a_\epsilon(u - u_N, u - u_N) = a_\epsilon(u - u_N, u - \Pi u)$ . We thus find by the Cauchy-Schwarz inequality that

$$(4.3) \quad |u - u_N|_{H^1} \leq \kappa |u - \Pi u|_{H^1} + \kappa \epsilon^{-1} |u - \Pi u|_{L^2}.$$

Hence (4.1) follows from the interpolation inequalities as in Lemma 2.3.  $\square$

**THEOREM 4.2** ( $L^2$ - error). *Assume only that (H0)-(2.6), (H1)-(2.7) hold. Let  $u = u^\epsilon$  be the exact solution of (1.3), and  $u_N$  the solution of (2.10), and let  $f$  be smooth on  $\bar{\Omega}$  satisfying (1.2). Then there exist positive constants  $\lambda$  and  $\kappa$  independent of  $\epsilon$ ,  $h_1$ ,  $h_2$  such that*

$$(4.4) \quad |u - u_N|_{L^2(\Omega)} \leq \kappa \begin{cases} \bar{h} + h_2^2 \epsilon^{-1/2} & \text{if } \epsilon \leq \lambda h_2^2, h_2 \leq \kappa h_1, \\ \bar{h} & \text{if } \lambda h_2^2 \leq \epsilon \leq \kappa h_2, \sqrt{2} h_1 \leq h_2 \leq \kappa h_1. \end{cases}$$

PROOF. Set  $\lambda = 1/(2\kappa_1) = \kappa_2$ , where  $\kappa_1$ ,  $\kappa_2$  are as in (H3)-(3.30), (H4)-(3.45). From (4.2), we have for all  $v \in V_N$ ,

$$(4.5) \quad a_\epsilon(u_N - \Pi u, v) = a_\epsilon(u - \Pi u, v) = \epsilon((u - \Pi u, v)) - ((u - \Pi u)_x, v).$$

By the uniqueness of solutions, we write

$$(4.6a) \quad u_N - \Pi u = v_N^1 + v_N^2,$$

where

$$(4.6b) \quad a_\epsilon(v_N^1, v) = \epsilon((u - \Pi u, v)), \quad \forall v \in V_N,$$

$$(4.6c) \quad a_\epsilon(v_N^2, v) = -((u - \Pi u)_x, v), \quad \forall v \in V_N.$$

From equation (4.6b) with  $v = v_N^1$ , we then easily find that

$$(4.7) \quad |v_N^1|_{H^1} \leq |u - \Pi u|_{H^1}.$$

If  $\epsilon \leq \lambda h_2^2$ ,  $h_2 \leq \kappa h_1$ , then from Theorem 3.1 applied to equation (4.6c) with  $f = -(u - \Pi u)_x$ , we find

$$(4.8) \quad |v_N^2|_{L^2} \leq \kappa |f|_{L^2} \leq \kappa |u - \Pi u|_{H^1}.$$

The first estimate in (4.4) follows from the interpolation inequality (2.11b) observing that due to the Poincaré inequality, (4.6), (4.7) and (4.8), we have

$$(4.9) \quad \begin{aligned} |u - u_N|_{L^2} &\leq |u - \Pi u|_{L^2} + |u_N - \Pi u|_{L^2} \\ &\leq |u - \Pi u|_{L^2} + \kappa |v_N^1|_{H^1} + |v_N^2|_{L^2} \leq \kappa |u - \Pi u|_{H^1}. \end{aligned}$$

If  $\lambda h_2^2 \leq \epsilon \leq \kappa h_2$ ,  $\sqrt{2}h_1 \leq h_2 \leq \kappa h_1$ , i.e. quasi-uniform elements. The second estimate in (4.4) similarly follows from Theorem 3.2 with  $f = -(u - \Pi u)_x$  applied to equation (4.6c) again:

$$(4.10) \quad |u - u_N|_{L^2} \leq \kappa |u - \Pi u|_{H^1} \leq \kappa (\bar{h} + h_2^2 \epsilon^{-1/2}) \leq \kappa \bar{h}.$$

□

REMARK 4.1. From Theorem 4.1 and Theorem 4.2, we find that for the new scheme (2.10) to be effective, we should require the space mesh to be of order  $\bar{h} = o(\epsilon^{1/2})$  in the  $H^1$  approximation and of order  $h_1$  small,  $h_2 = o(\epsilon^{1/4})$  in the  $L^2$  approximation.

REMARK 4.2. In Figure 2, we present one numerical example. The classical scheme (CS) in (a) shows wild oscillations in the convective direction ( $x$ -direction due to  $-u_x^\epsilon$  here) due to the discretization errors of ordinary boundary layers at  $x = 0$ . On the other hand, the new scheme (NS) in (b), which incorporates the boundary layer element (BLE)  $\phi_0$ , see (c), as defined in (2.3), captures the singularities due to the boundary layers. Note that the approximate solution  $u_N$  is composed of (c) OBL

and (d) classical elements. Extensive numerical simulations (with various boundary conditions) appear in [1] - [5] and elsewhere.

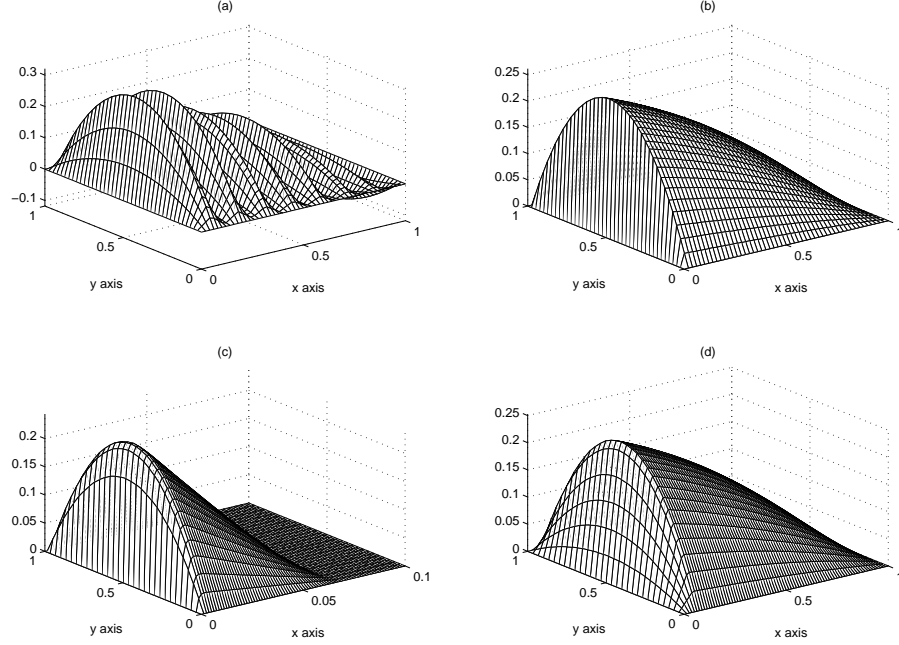


FIGURE 2. Classical Scheme (CS) v.s. New Scheme with BLE  $\phi_0$  (NS) for  $-\epsilon \Delta u^\epsilon - u_x^\epsilon = e^x(1 - e^{-y})(1 - e^{-(1-y)})$ ,  $\epsilon = 10^{-3}$ ,  $M = 15$ ,  $N = 20$  : (a) (CS), (b) (NS), (c) Boundary layer elements (BLE) (zoomed near  $x = 0$ ) and (d) Classical elements (hat functions); (b) = (c) + (d).



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## CURRICULUM VITAE

### Research Interests

- Partial differential equations, Numerical analysis and Scientific Computing
- Asymptotic analysis: singular perturbations, boundary layer analysis and its numerical applications
- Numerical methods: finite elements method, mixed elements method, continuation method, multilevel method, spectral elements method
- (Geophysical) Fluid dynamics, Bifurcations

### Education

- Indiana University, Bloomington, Indiana, USA  
Ph.D., Mathematics, May 2006 (advisor: Prof. Roger Temam)  
Ph.D. Minor, Scientific Computing, May 2006
- Pohang University of Science and Technology, Pohang, Korea  
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### Employment

- 2000 Aug - present: Teaching and Research assistant, Indiana University
- 2003 Fall, 2004 Fall, 2005 Fall semesters: NSF Research assistant, Indiana University
- 1997 - 2000: System and database manager, client/server programmer, Prudential (life insurance company) of Korea, Seoul, Korea
- 1993 - 1996: Korea army (for military service), Daejeon, Korea

### Publications

- [1] C. Jung, Numerical approximation of two-dimensional convection-diffusion equations with boundary layers. *Numerical Methods for Partial Differential Equations*, 21(2005), no. 3, pp. 623-648.
- [2] C. Jung, Numerical approximation of convection-diffusion equations in a channel using boundary layer elements. *Applied Numerical Mathematics*. 56(2006), pp 756-777.
- [3] C. Jung and R. Temam, Numerical approximation of two-dimensional convection-diffusion equations with multiple boundary layers. *International Journal of Numerical Analysis and Modeling*. 2(2005), no. 4, pp 367-408.
- [4] C. Jung and R. Temam, On parabolic boundary layers for convection-diffusion equations in a channel: Analysis and Numerical applications. *Journal of Scientific Computing*. To appear.
- [5] C. Jung and R. Temam, Construction of boundary layer elements for singularly perturbed convection-diffusion equations and  $L^2$ - stability analysis. *Submitted to SIAM Journal on Numerical Analysis*, 2005.
- [6] C. Jung and R. Temam, Asymptotic analysis for singularly perturbed convection-diffusion equations with a turning point. *Submitted to Journal of Mathematical Physics*, 2006.